LECTURE 21: SUMMARY

We continued our discussion of examples of metric spaces. First, we corrected our definition of the "British Rail" metric. Recall that we had defined the distance between x and y in $\mathbb{R}^n$ as $d(x, y) := |x| + |y|$; in other words, to travel from x to y, you have to go through the origin. This almost defines a metric, but not quite: $d(x, x)$ might be nonzero. Kelvin suggested that the problem is that, to go from a city to itself, one doesn't need to travel through London. So, we made the following (corrected) definition of the British Rail metric:

$$d(x, y) := \begin{cases} |x| + |y| & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

Having made this correction, we continued our discussion of examples of metric spaces. We left off at the end of last lecture with the following important example:

(vii) The space $\mathcal{C}_{[0,1]} := \{f : [0, 1] \to \mathbb{R}, \text{ a continuous function}\}$, with the metric

$$d(f, g) = \max_{t \in [0,1]} |f(t) - g(t)|.$$

First, note that this is well-defined, since the extreme value theorem guarantees that a continuous function attains a maximum on a closed interval. (This is an important point; for example, $\max_{t \in (0,1)} |f(t) - g(t)|$ would not be a metric on $\mathcal{C}_{[0,1]}$. Can you explain why not?) As discussed, proving the triangle inequality is rather similar to the chessboard metric (example (iv) from last lecture).

(viii) Let

$$\mathcal{L}^p := \left\{ f : \mathbb{R} \to \mathbb{R} \text{ s.t. } \int |f|^p \text{ converges} \right\}$$

We make this into a metric space under the metric

$$d(f, g) := \left(\int |f - g|^p \right)^{1/p}$$

(ix) Similarly, we can treat the space of sequences, rather than of functions. Let

$$\ell^p := \left\{ (a_n) \text{ a sequence of real numbers} : \sum |a_n|^p \text{ converges} \right\}$$

This is a metric space with respect to the metric

$$d\Big((a_n), (b_n)\Big) := \left(\sum |a_n - b_n|^p \right)^{1/p}$$

(x) We concluded our examples with a discussion of a simple but useful metric space: $\mathbb{N}$ under the *Hamming metric*. To calculate the distance between two integers a and b, write each in binary

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notation. The distance between a and b is the number of (binary) digits in which a and b disagree. For example, to calculate the distance between 40 and 13, first write each in binary:

$$40 = \mathbf{1}01\mathbf{0}0\mathbf{0}$$
$$13 = \mathbf{0}01\mathbf{1}0\mathbf{1}.$$

The digits which disagree are highlighted; there are three of them. Thus,

$$d(40, 13) = 3.$$

To prove triangle inequality, we rewrote the definition of the Hamming metric in mathematical notation:

$$d(a, b) = \sum_{i=1}^{n} |a_i - b_i|$$

where $a = a_1 a_2 \cdots a_n$ and $b = b_1 b_2 \cdots b_n$ in binary notation. Triangle inequality now follows easily from the standard one.