LECTURE 22: SUMMARY

Today we started exploring point-set topology, the final topic of the term. To motivate the subject, we started by considering the following question. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. What does it do to subsets of $\mathbb{R}$? For example, what can one say about the image of an open interval? We quickly realized that the image must also be an interval. However, that's pretty much all one can say! The image might be a single point (e.g. the function $f(x) = x$ on any open interval); a closed interval ($f(x) = \sin x$ on any open interval longer than 2π); an open interval; or a half-open interval (can you generate an example?). So, continuous functions don't preserve openness of an interval.

If instead we look at the pre-image, the situation changes: by playing around, we discovered that the pre-image of an open interval is always a union of open intervals. For example, for the function $f(x) = x^2$, the pre-image of an open interval might be one or the union of two open intervals. It's even possible for the pre-image to be a countably infinite union of open intervals (can you come up with an example?). Thus, under pre-imaging, continuous functions almost preserve open intervals; the only flaw is that the pre-image might consist of more than one interval. This led us to call a subset of $\mathbb{R}$ *open* iff it is a union of finitely or countably many open intervals. Our observation above can now be stated quite simply: the pre-image (under a continuous function) of an open set is open. It turns out that the converse is also true:

Theorem 1. *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous* if and only if *$f^{-1}(A)$ is open for all open sets A.*

Suppose (X, d) is a metric space. What is the equivalent of an open interval on X? Even in the case of $\mathbb{R}^2$ under the standard (Euclidean) metric, it's not obvious what the appropriate analogy is, much less in the other examples of metric spaces we've discussed. The theorem above suggests an approach to this question: the set of open sets in a X should be preserved under pre-images of continuous functions (which are easy to define in a metric space). After a lot of work, people realize the appropriate analogue of open in an arbitrary metric space:

Definition. *Given a metric space (X, d). We say $\mathcal{O} \subseteq X$ is* open *iff for every point $p \in \mathcal{O}$, there is a neighbourhood of p entirely contained in $\mathcal{O}$.*

Here, the *neighbourhood of p of radius r*, denoted $\mathcal{N}_r(p)$, is defined to be all points of the space which are close to p:

$$\mathcal{N}_r(p) := \{x \in X : d(p, x) < r\}.$$

Thus we can rewrite the definition of open: $\mathcal{O} \subseteq X$ is open iff for every $p \in \mathcal{O}$, there exists $\delta > 0$ such that $\mathcal{N}_\delta(p) \subseteq \mathcal{O}$. For example, in $\mathbb{R}^2$ the interval $(0, 1) \times \{0\}$ is not open. By contrast, the strip $(0, 1) \times \mathbb{R}$ is open. (Draw pictures of these two examples.)

Given an arbitrary metric space, we can now at least produce examples of open sets:

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Theorem 2. *Given metric space (X, d), $x \in X$, and $r > 0$. Then the neighbourhood $\mathcal{N}_r(x)$ is open.*

Proof. Pick $y \in \mathcal{N}_r(x)$. Let

$$\delta := r - d(x, y).$$

By definition of $\mathcal{N}_r(x)$, we see that $\delta > 0$. I now claim that

$$n_\delta(y) \subseteq \mathcal{N}_r(x). \tag{*}$$

To see this, pick any $p \in n_\delta(y)$. Then by definition,

$$d(p, y) < \delta = r - d(x, y).$$

By triangle inequality, it follows that

$$d(p, x) \leq d(p, y) + d(y, x) < r.$$

This proves (*), and we conclude that $\mathcal{N}_r(x)$ is open. $\square$

We finished lecture with an important result about combining open sets.

Theorem 3. *Arbitrary unions of open sets are open.*

Proof. Given a collection $\{\mathcal{O}_\alpha\}$ of open sets, let

$$\mathcal{A} := \bigcup_\alpha \mathcal{O}_\alpha.$$

We wish to show that $\mathcal{A}$ is open. Pick $p \in \mathcal{A}$. Then $p \in \mathcal{O}_\beta$ for some β. Since $\mathcal{O}_\beta$ is open, there exists $\delta > 0$ such that $\mathcal{N}_\delta(p) \subseteq \mathcal{O}_\beta$. This immediately implies that $\mathcal{N}_\delta(p) \subseteq \mathcal{A}$, and we conclude that $\mathcal{A}$ is open. $\square$