LECTURE 23: SUMMARY

Last time, we discussed the notion of open sets in a general metric space. We proved that neighbourhoods are open, and that arbitrary unions of open sets are open. What about intersections? After a bit of thought, we saw that the intersection of a bunch of open sets need not be open; for example,

$$\bigcap_{n\in\mathbb{N}}\left(-\frac{1}{n},\frac{1}{n}\right)=\{0\}.$$

If we're slightly less greedy, we can still prove something about intersections:

Theorem 1. *The intersection of finitely many open sets is open.*

Proof. Suppose $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_N$ are open in the metric space (X, d), and set

$$\mathcal{A} := \bigcap_{n\le N}\mathcal{O}_n.$$

Pick $p \in \mathcal{A}$; we wish to find a neighbourhood of p which is entirely contained in $\mathcal{A}$.

By definition of $\mathcal{A}$, we know that $p \in \mathcal{O}_n$ for each $n \le N$; since $\mathcal{O}_n$ is open, there exists $\delta_n > 0$ such that $\mathcal{N}_{\delta_n}(p) \subseteq \mathcal{O}_n$ for every $n \le N$. Let

$$\delta := \min\{\delta_n : n \le N\}.$$

Then $\delta > 0$, and $\mathcal{N}_\delta(p) \subseteq \mathcal{O}_n$ for every n. (Why?) It follows that $\mathcal{N}_\delta(p) \subseteq \mathcal{A}$ as well. This concludes the proof. $\square$

This concludes our discussion of open sets, and we move on to their counterparts, the closed sets. Before defining what a closed set is, we discussed the notion of limit point. Given a metric space (X, d) and a subset $S \subseteq X$, we say $\ell \in X$ is a *limit point of* S iff it is the limit of some sequence of distinct points of S. In other words, ℓ is a limit point iff there exists a sequence (s_n) of points in S such that (i) $s_m \ne s_n$ whenever $m \ne n$, and (ii) $\lim\limits_{n\to\infty} s_n = \ell$.

Note that a limit point of S might live in S itself, or might not live in S. For example, consider $\mathbb{R}$ with respect to the usual metric. Then 1 is a limit point of $[0, 1]$ (it is the limit of $1-\frac{1}{n}$). 1 is also a limit point of $(0, 1)$. Similarly, both 0 and $\sqrt{2}$ are limit points of $\mathbb{Q}$. What are all the limit points of $\mathbb{Q}$?

We've been talking about limits without precisely defining them in an abstract metric space. Fortunately, the definition is exactly what you might imagine. Given a sequence (a_n) of points in a metric space (X, d), we say

$$\lim_{n\to\infty} a_n = A \quad\Longleftrightarrow\quad \forall\epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } d(a_n, A) < \epsilon \text{ whenever } n > N.$$

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In practice, it is not always easy to verify whether a point is a limit point of a given set. For example, above we decided that $\sqrt{2}$ is a limit point of $\mathbb{Q}$. But how do we know for sure? Can you tell me a sequence of rationals which converges to $\sqrt{2}$? What's the millionth term of this sequence? Fortunately, there is an easier way to check limit points.

Theorem 2. *α is a limit point of S iff every neighbourhood of α contains some point of S (other then α itself).*

In other words, α is a limit point of S iff for each $\epsilon > 0$, there exists $x \in \mathcal{N}_\epsilon(\alpha) \cap S$ such that $x \neq \alpha$. For example, in the space $\mathbb{R}$ (with respect to the usual metric), $\sqrt{2}$ is a limit point of $\mathbb{Q}$ because every open interval contains a rational number.

Proof. As usual, we prove the two directions separately.

($\Longrightarrow$) Given $\epsilon > 0$. Since α is a limit point of S, there exists a sequence of points (s_n) in S which converge to α. Thus, there exists N such that $s_n \in \mathcal{N}_\epsilon(\alpha)$ for all $n > N$. In particular, the two distinct points s_{N+1} and s_{N+2} belong to $\mathcal{N}_\epsilon(\alpha)$. One of these two must be distinct from α, and the claim follows.

($\Longleftarrow$) There exists $s_1 \in S$ such that $0 < d(s_1, \alpha) < 1/2$. Let $\epsilon_1 := d(s_1, \alpha)$. There exists $s_2 \in \mathcal{N}_{\epsilon_1/2}(\alpha) \cap S$ such that $s_2 \neq \alpha$. It follows that $d(s_2, \alpha) \leq \epsilon_1/2$, and that $s_2 \neq s_1$. Set $\epsilon_2 := d(s_2, \alpha)$. Proceeding in this way, we construct a sequence of distinct points (s_n), which satisfy

$$d(s_n, \alpha) < \frac{1}{2^n}$$

for all n. It follows that (s_n) converges to α, whence α is a limit point of S. $\square$