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MATB43: ANALYSIS

Midterm Exam (due by 4pm SHARP on Wednesday, February 27th, in the appropriate dropbox on the 4th floor of the IC building)

INSTRUCTIONS: Please print and attach this page as the first page of your submitted problem set. You may freely refer to the textbook (Kolmogorov-Fomin) or to your lecture notes. However, you may NOT use the internet in any way (even to look up definitions), nor are you allowed to consult with any human regarding the problems. If you are unsure whether something is allowed, please check with me first.

PROBLEM	MARK
M.1	
M.2	
M.3	
M.4	
M.5	
M.6	
M.7	
M.8	
Total	

Please read the following statement and sign below:

I understand that I am not allowed to use the internet to assist (in any way) with this midterm; I am also not allowed to interact with any human in any way about these problems.

SIGNATURE:______________________________

Midterm Exam

M.1 Given $a, b \in \mathbb{R}$, recall the notations

$$\begin{array}{cc} (a,b) := \{x \in \mathbb{R} : a < x < b\} & [a,b] := \{x \in \mathbb{R} : a \le x \le b\} \\ [a,b) := \{x \in \mathbb{R} : a \le x < b\} & (a,b] := \{x \in \mathbb{R} : a < x \le b\} \\ (a,\infty) := \{x \in \mathbb{R} : x > a\} & (-\infty,b) := \{x \in \mathbb{R} : x < b\} \\ [a,\infty) := \{x \in \mathbb{R} : x \ge a\} & (-\infty,b] := \{x \in \mathbb{R} : x \le b\}. \end{array}$$

An *interval* is any set which is either all of $\mathbb{R}$, or else can be written in one of the eight forms above.

(a) Prove that $(0,1) \sim (a,b)$ for any real numbers $a < b$, by giving an *explicit* bijection. [*You must also prove that your map is a bijection!*]

(b) Prove that for any real number a we have $(0,1) \sim (a,\infty)$, by giving an *explicit* bijection.

(c) Prove that for any real b, we have $(0,1) \sim (-\infty,b)$. [*Hint: use part (b)*]

(d) Prove that $(0,1) \sim \mathbb{R}$ by giving an *explicit* bijection.

(e) Prove that if A is any infinite set and $x \notin A$, then $A \sim A \cup \{x\}$. [*Hint: we proved in lecture that A must contain a countable subset.*]

(f) Prove that $(0,1)$, $[0,1)$, $(0,1]$, and $[0,1]$ are all equivalent. [*Hint: use part (e) and transitivity.*]

(g) Is it true that any two non-empty intervals are equivalent? State and prove a (correct!) theorem along these lines.

M.2 Recall that given a set A, the power set $\mathcal{P}(A)$ is the set consisting of all subsets of A. Given two sets A and B, define B^A to be the set of all functions $f : A \to B$. (The next exercise gives some justification to this notation.) Prove that $\mathcal{P}(A) \sim \{0,1\}^A$ for any $A \neq \emptyset$. [*Hint: prove this separately for finite and infinite A.*]

M.3 Let $A_n := \{1, 2, \ldots, n\}$, and let B be any nonempty set. Define $\prod_{a \in A_1} B := B$, and for each $n \ge 2$ define $\prod_{a \in A_n} B := B \times \prod_{a \in A_{n-1}} B$. (The definition of $A \times B$ is given in problem M.8 below.) Prove that

$$B^{A_3} \sim \prod_{a \in A_3} B.$$

Recall that an binary relation $\prec$ on a set A is an *order* on A iff it satisfies three properties:

(i) Antisymmetry: $a \not\prec a$ for all $a \in A$

(ii) Transitivity: if $a \prec b$ and $b \prec c$, then $a \prec c$

(iii) Comparability: for any $a, b \in A$, either $a = b$, $a \prec b$, or $b \prec a$.

M.4 In property (iii) above – comparability – the 'or' is not exclusive: the property leaves open the possibility that two (or even all three) of the conditions hold simultaneously for some elements a and b. Prove that in fact, the 'or' *must* be exclusive. In other words, given an order $\prec$ on a set A and any two elements $a, b \in A$, prove that *exactly* one of the relations $a = b$, $a \prec b$, or $b \prec a$ holds.

M.5 In this problem, you will show that all three conditions of being an order are necessary; none of them can be derived from the others.

(a) Give an example of a set A and a binary relation $\prec$ on A which satisfies properties (i) and (ii), but not (iii).

(b) Give an example of a set A and a binary relation $\prec$ on A which satisfies properties (i) and (iii), but not (ii).

(c) Give an example of a set A and a binary relation $\prec$ on A which satisfies properties (ii) and (iii), but not (i).

M.6 First, recall some notation from our proof of the Cantor-Bernstein theorem. Given sets A and B and injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$, set $A_0 = A$, $B_0 = B$, $A_{n+1} = g(B_n)$, and $B_{n+1} = f(A_n)$ for all n. Further, set $A_n^* = A_n - A_{n+1}$ and $B_n^* = B_n - B_{n+1}$. Finally, let

$$\widetilde{A} = \bigcup_{n\geq 0} A_n^* \qquad \widetilde{B} = \bigcup_{n\geq 0} B_n^* \qquad \overline{A} = \bigcap_{n\geq 0} A_n \qquad \overline{B} = \bigcap_{n\geq 0} B_n$$

Let $A = \mathbb{Z} + \frac{1}{2} := \{n + \frac{1}{2} : n \in \mathbb{Z}\}$ and $B = \mathbb{Z}$, and consider $f : A \to B$ and $g : B \to A$ defined by $f(x) = 2x$ and $g(x) = 3x - \frac{5}{2}$.

(a) Give an explicit description of the sets A_n and B_n (in terms of n).

(b) Give an explicit description of $\widetilde{A}$ and $\widetilde{B}$.

(c) Give an explicit description of $\overline{A}$ and $\overline{B}$.

M.7 In class we defined an injection $[0, 1) \hookrightarrow \{0, 1\}^{\mathbb{N}}$ by sending $x \in [0, 1)$ to a recursively-defined function f_x. [Recall: we defined $f_x(1)$ to be 0 if $x \in [0, 1/2)$, and 1 otherwise. In the former case we set $a_1 = 0$ and $b_1 = 1/2$; in the latter, $a_1 = 1/2$ and $b_1 = 1$. Next, for each $n \geq 2$, define $f_x(n)$ to be 0 if $x \in [a_{n-1}, \frac{a_{n-1}+b_{n-1}}{2})$, and 1 otherwise. In the former case, set $a_n = a_{n-1}$ and $b_n = \frac{a_{n-1}+b_{n-1}}{2}$; in the latter, set $a_n = \frac{a_{n-1}+b_{n-1}}{2}$ and $b_n = b_{n-1}$.]

(a) Describe (with proof!) the function $f_{2/5}(n)$. [*Hint: start by computing $f_{2/5}(n)$ for $n = 1$ to 7.*]

(b) Prove that $\lim_{N\to\infty} \sum_{n=1}^{N} \frac{f_x(n)}{2^n} = x$. [In other words, prove that for any $\epsilon > 0$, there exists a positive integer N_0 such that for all $N \geq N_0$, $\left| \sum_{n=1}^{N} \frac{f_x(n)}{2^n} - x \right| < \epsilon$.

M.8 Recall that $A \times B := \{(a, b) : a \in A, b \in B\}$, where (a, b) denotes an ordered pair (not an interval). Prove that the Generalized Continuum Hypothesis implies that $A \times A \sim A$ for any infinite set A. [In fact, this can be proved *without* the GCH! In your proof, feel free to use the axiom of choice and the well-ordering theorem.]