# GROUPS AND SYMMETRY: LECTURE 2

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We began by recalling from last lecture our definition of congruence:

**Definition.** *Two sets* $A, B \subseteq \mathbb{R}^2$ *are* congruent*, denoted* $A \cong B$*, iff there exists a rigid motion* $\phi$ *such that* $\phi(A) = B$*.*

In this definition, recall that a *rigid motion* is a special type of function mapping the plane to itself, which captures the intuitive notion of moving shapes around the plane. Here's the formal definition we came up with last time:

**Definition.** *A* rigid motion *(of the plane) is a function* $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ *which preserves distances, i.e. for all* $X, Y \in \mathbb{R}^2$ *we have*

$$d\Big(\phi(X), \phi(Y)\Big) = d\Big(X, Y\Big).$$

Here we are using the standard Euclidean distance $d(X, Y) = |X - Y|$.

One thing I pointed out is that although we've been using the term *rigid motion*, it's conventionally called an *isometry*. Where does this word come from? As is often the case, Greek: $\iota\sigma o\zeta$ (*isos*) means equal, and $\mu\epsilon\tau\rho o\nu$ (*metron*) means measure. So even though it sounds fanciful, 'isometry' is a pragmatic name. We will use 'isometry' and 'rigid motion' interchangeably from now on.

## 1. Isometries

Defining congruence in terms of isometry is nice, but useless if we don't know anything about isometries. What can we say about the isometries of the plane? Last time, while trying to come up with a good definition of congruence, we generated three examples of isometries:

(1) Translations, e.g. the map which pushes every point of the plane 2 units to the right and 1 unit down;
(2) Rotations, e.g. the map which rotates every point of the plane by an angle of $\pi/7$ around the point $(3, 5)$; and
(3) Reflections, e.g. the map which flips every point of the plane over the line $y = 4x + 1$.

As Dinu observed in lecture, these three isometries can be composed with each other to generate many other isometries. For example, we created a complicated isometry $h : \mathbb{R}^2 \to \mathbb{R}^2$ whose action is defined by a sequence of operations: first, rotate around the origin by $2\pi/7$, then translate by $(2, 3)$, then flip over the line $y = 2x - 1$, then rotate by $\pi/17$ around the point $(1, 3)$. This crazy composition of isometries must itself be an isometry, since at each stage we're not stretching or shrinking distances between any two points.

At this point, many of you suspected that every isometry can be formed this way, i.e. as a composition of the three basic isometries listed above. This is true, but it turns out that something much stronger and more bizarre is the case.

**Theorem 1.** *Every isometry of the plane is either a translation, a rotation, or a glide reflection.*

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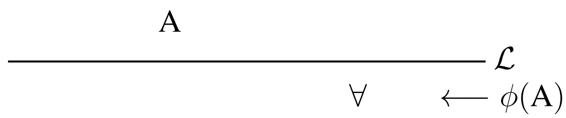
Before we can discuss this theorem further, we need to define a *glide reflection*: it is the isometry which flips every point over a given line $\mathcal{L}$, and then translates *parallel* to $\mathcal{L}$. Thus, it is a generalization of a reflection (a reflection is simply a glide reflection with glide 0). We can picture the action of an isometry $\phi$ by taking a random set of points (say, in the shape of an A) and then looking at where those points end up after applying $\phi$. Here's the picture we get if $\phi$ is a glide reflection and $\mathcal{L}$ is a horizontal line:




To see what this would look like if $\mathcal{L}$ were some non-horizontal line, just tilt your head! More generally, we discussed that rather than moving the plane, one can imagine an isometry in terms of moving yourself and changing your point of view. This point might seem a bit silly, but it is quite subtle and will play an important role in the coming lectures.

One consequence of Theorem 1 is that our crazy isometry $h$ from before must be a translation, a rotation, or a glide reflection. In fact, as Jay observed, of these three $h$ can only be a glide reflection; translations and rotations don't flip left-handed and right-handed objects, while $h$ does. That $h$ is a glide reflection is highly non-obvious, and I defy the reader to find the line with respect to which $h$ acts.

We also discussed some related subjects: Pasteur's discovery of left- and right-handed symmetry, the theorem of crystallographic restriction, and wallpaper designs. We'll return to these topics in more detail in the future.

We finished lecture with a question. It's not too hard to write down an algebraic (ie symbolic) definition of translation (in terms of vector addition, say). But what about rotation? For example, can you write down in symbols a function which rotates every point of the plane by $\pi/6$? We'll start with this question on Friday.

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