GROUPS AND SYMMETRY: LECTURE 4

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Last lecture, we came up with short algebraic descriptions of a few fundamental plane isometries; the goal of this lecture is to explore some more general isometries. We started by reviewing the three isometries we dealt with last lecture. They are most conveniently described in the language of complex numbers (i.e. viewing the plane not as $\mathbb{R}^2$, but as $\mathbb{C}$).

(1) Translation by a is the function defined by $T_a(z) = z + a$ for all $z \in \mathbb{C}$.

(2) Rotation by α radians (clockwise) around the origin is the function defined by $R_\alpha(z) = e^{i\alpha}z$ for all $z \in \mathbb{C}$.

(3) Reflection across the x-axis is the function defined by $\rho(z) = \bar{z}$, where $\bar{z}$ denotes the complex conjugate of z.

What about other isometries?

We started by discussing what a rotation around $C \neq 0$ might look like. Eric suggested that we can rotate around C in three steps: first, translate the plane until C ends up where the origin used to be (thus making C play the role of the origin); next, apply R_α to rotate every point around the origin (which, at the moment, is actually C); finally, translate the plane so that C gets back to its original location. In terms of algebra, Eric is suggesting that the isometry

$$T_C \circ R_\alpha \circ T_{-C}$$

rotates the plane by α (counterclockwise) around the point C. Make sure you understand why T_C is on the left, and T_{-C} is on the right.

Here's a physical way to grasp Eric's algorithm. Forget about the usual way of labeling the plane for a moment. Instead, sit down at a desk, and put a blank piece of paper in front of you on the desk. Lean over it, so that your eyes are directly above the middle of the paper. Mark the spot on the paper directly beneath your eyes, and call it O. Our operation R_α is to grab the paper and rotate it α radians counterclockwise, keeping the point directly beneath your eyes fixed. Don't apply this operation yet. Instead, pick a random point C somewhere else on the paper. We want to rotate around C. Here's Eric's algorithm: slide the paper (without moving your own head) until the point C is directly beneath your eyes; now rotate by some amount; now slide the paper back so that C is where it used to be. The result is that every point on the paper is rotated by α around C from where it used to be.

Note that if $C \neq 0$, the isometry $T_C \circ R_\alpha \circ T_{-C}$ is different from R_α, since the former fixes the point C while the latter doesn't. It follows that unless $C = 0$,

$$T_C \circ R_\alpha \neq R_\alpha \circ T_C.$$

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Thus, composition of isometries behaves a bit differently from multiplication of numbers – the order in which you compose to functions matters! This is reminiscent of matrix multiplication. It's also a bit annoying, because it means we can't simply complicated isometries like $T_C \circ R_\alpha \circ T_{-C}$ in the most obvious way.

Dan proposed a different approach to simplifying the isometry

$$\phi := T_C \circ R_\alpha \circ T_{-C}.$$

His idea was that ϕ rotates the plane by α, and R_α rotates the plane by α, so up to a translation they should look the same. In other words, Dan claimed that we should be able to write $\phi = T_h \circ R_\alpha$. Dan's idea implies more: that given α and k, there should exist ℓ and θ such that $R_\alpha \circ T_k = T_\ell \circ R_\theta$. Is this true? Let's do a quick calculation to find out.

Pick any $z \in \mathbb{C}$. We have

$$\begin{aligned} R_\alpha \circ T_k(z) &= R_\alpha(z+k) \\ &= e^{i\alpha}(z+k) \\ &= e^{i\alpha}z + e^{i\alpha}k \\ &= T_{e^{i\alpha}k}(e^{i\alpha}z) \\ &= T_{R_\alpha(k)}\big(R_\alpha(z)\big) \\ &= T_{R_\alpha(k)} \circ R_\alpha(z). \end{aligned}$$

In other words, for every $z \in \mathbb{C}$ the two isometries $R_\alpha \circ T_k$ and $T_{R_\alpha(k)} \circ R_\alpha$ agree. By definition, this means they are the same function:

$$R_\alpha \circ T_k = T_{R_\alpha(k)} \circ R_\alpha.$$

So Dan's idea is right! This also implies that his initial instinct, that we should be able to write $\phi = T_h \circ R_\alpha$, is correct as well! Indeed, using our above calculation we have

$$\begin{aligned} \phi &= T_C \circ R_\alpha \circ T_{-C} \\ &= T_C \circ T_{R_\alpha(-C)} \circ R_\alpha \\ &= T_{C+R_\alpha(-C)} \circ R_\alpha. \end{aligned}$$

Thus, we have shown that *every* rotation can be written in the simple form $T_h \circ R_\alpha$ for some h and α. Is the converse true? Turns out that the answer is yes, as you will prove on your next assignment. In other words, the following holds:

Proposition 1. *An isometry ϕ is a nontrivial rotation of the plane* if and only if $\exists h \in \mathbb{C}$ *and a real $\alpha \neq 0$ such that $\phi = T_h \circ R_\alpha$.*

This statement has some nice consequences. We'll explore these, as well as glide reflections, in Friday's lecture.

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