GROUPS AND SYMMETRY: LECTURE 7

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Recall that $\mathcal{G}$ denotes the set of all plane isometries. We also decided that writing $f \circ g$ to denote function composition is tedious; henceforth, this will be denoted by fg. (This notation isn't as ambiguous as it may seem at first glance – how else could it be interpreted?)

Recall that last lecture, we proved most of the following.

Lemma 1. *Given $\phi \in \mathcal{G}$, there exist $h \in \mathbb{C}$, $\alpha \in [0, 2\pi)$, and $j \in \{0, 1\}$ such that*

$$\phi = T_h R_\alpha \rho^j.$$

Moreover, h, α, and j are uniquely determined by ϕ.

Here are the main steps. Most of these were completed the previous lecture, so I suppress details. However, a few things changed in Steps 3 and 4.

Proof.

STEP 1: Renormalize ϕ so that it fixes the origin.

More precisely, define $f := T_{-\phi(0)}\phi$. It is easy to verify that $f(0) = 0$ and that $f \in \mathcal{G}$. It follows that $|f(X)| = |X|$ for all $X \in \mathbb{R}^2$. //

STEP 2: Dot products are preserved: for all $X, Y \in \mathbb{R}^2$, we have $f(X) \cdot f(Y) = X \cdot Y$.

Since $f \in \mathcal{G}$, $|f(X) - f(Y)|^2 = |X - Y|^2$. Expanding and simplifying both sides yields the claim. //

STEP 3: f is linear: $f(\alpha X + \beta Y) = \alpha f(X) + \beta f(Y)$ for all $\alpha, \beta \in \mathbb{R}$ and $X, Y \in \mathbb{R}^2$.

We do this in two steps. First, consider $|f(\alpha X) - \alpha f(X)|^2$. Writing this in terms of the dot product, expanding, applying Step 2, and simplifying shows that $f(\alpha X) = \alpha f(X)$. A similar argument shows that $f(X + Y) = f(X) + f(Y)$, and the claim immediately follows. //

STEP 4: $f = R_\alpha \rho^j$ for some $\alpha \in [0, 2\pi)$ and $j \in \{0, 1\}$.

First, note that by linearity,

$$f(a + bi) = af(1) + bf(i) \tag{1}$$

for any $a, b \in \mathbb{R}$. Next, observe that $|f(1)| = |f(i)| = 1$, whence we can write $f(1) = e^{i\alpha}$ and $f(i) = e^{i\beta}$ for some $\alpha, \beta \in [0, 2\pi)$. Moreover, Step 2 implies that $f(1) \perp f(i)$, so $|\beta - \alpha| = \pi/2 + 2\pi k$ for some $k \in \mathbb{Z}$. Thus $\beta = \alpha \pm \pi/2 + 2\pi k$, and it follows that

$$f(i) = ie^{i\alpha} \quad \text{or} \quad f(i) = -ie^{i\alpha}.$$

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In the former case, (1) implies that $f = R_\alpha$; in the latter case, (1) implies that $f = R_\alpha \rho$. The claim is proved. //

STEP 5: Existence and Uniqueness

Putting together Steps 1 and 4, we see that

$$\phi = T_{\phi(0)} f = T_{\phi(0)} R_\alpha \rho^j.$$

We have thus proved that it's *possible* to express any isometry ϕ in the form $\phi_h R_\alpha \rho^j$. We now prove uniqueness. Suppose

$$T_h R_\alpha \rho^j = T_\ell R_\beta \rho^k,$$

where $h, \ell \in \mathbb{C}$, $\alpha, \beta \in [0, 2\pi)$, and $j, k \in \{0, 1\}$. We wish to show that $h = \ell$, $\alpha = \beta$, and $j = k$.

Jay offered the following argument. First, observe that the two isometries are equal iff

$$T_h R_\alpha \rho^j(z) = T_\ell R_\beta \rho^k(z)$$

for all $z \in \mathbb{C}$. Taking $z = 0$ implies that $h = \ell$. It follows that

$$R_\alpha \rho^j(z) = R_\beta \rho^k(z)$$

for all $z \in \mathbb{C}$. Taking $z = 1$ implies that $e^{i\alpha} = e^{i\beta}$, whence $\alpha = \beta + 2\pi k$ for some $k \in \mathbb{Z}$. Since $\alpha, \beta \in [0, 2\pi)$, we deduce that $\alpha = \beta$. Thus, we conclude that

$$\rho^j = \rho^k.$$

Without loss of generality, say $j \leq k$, so that $k - j = 0$ or 1. Then

$$\rho^{k-j}(z) = z$$

for all $z \in \mathbb{C}$. Taking $z = i$ yields $k - j = 0$, and hence, that $j = k$. Uniqueness is proved! □

In fact, during the course of the proof we figured out more precisely what h, α, and j are in the statement of the Lemma. It is a good exercise to describe them precisely in terms of ϕ.

Let's take stock of what we've proved so far about the structure of $\mathcal{G}$ thus far.

(1) Every $\phi \in \mathcal{G}$ can be written in the form $T_h R_\alpha$ or $T_h R_\alpha \rho$.
(2) $T_h R_\alpha$ is a rotation whenever $\alpha \neq 0$.

Combining these, we see that any isometry ϕ is either a translation (if $\alpha = 0$ and $j = 0$), a rotation (if $\alpha \neq 0$ and $j = 0$), or has the form $\phi = T_h R_\alpha \rho$. We will show that every isometry of the last form must be a glide reflection, thus proving the following fundamental result:

Theorem 2 (Classification of Plane Isometries)**.** *Every plane isometry is either a rotation, a translation, or a glide reflection.*

As mentioned above, it suffices to prove that $T_h R_\alpha \rho$ is a glide reflection. Why is this true? It's entirely unclear at first glance (and I challenge the reader to see this in a picture). As with the case of the rotation, we build intuition by going in reverse: we'll prove that any glide reflection can be written in this form.

Let's start with some notation, to make our lives easier. Given a line $\mathcal{L}$ and a real number a, let $\gamma_{\mathcal{L},a}$ denote the glide reflection along $\mathcal{L}$ with displacement[1] a. By the Lemma, we know we can express $\gamma_{\mathcal{L},a}$ as a composition of T's, R's, and (possibly) ρ. How? Following an idea of Steph and Kishan from a couple lectures ago, we first rotate the plane until $\mathcal{L}$ is horizontal, then translate the plane up or down until $\mathcal{L}$ becomes the x-axis. In other words, we can find $\alpha \in [0, 2\pi)$ and $b \in \mathbb{R}$ such that

$$T_{bi}R_{\alpha}(\mathcal{L}) = x\text{-axis}.$$

Next, we reflect across the x-axis, and glide by a; finally, we undo our original vertical shift / rotation to bring the x-axis back to where $\mathcal{L}$ was originally. In other words, we have

$$\gamma_{\mathcal{L},a} = R_{-\alpha}T_{-bi}T_a\rho T_{bi}R_{\alpha}.$$

This looks complicated, but using the rules we've developed for switching the orders of primitive isometries, we can rewrite this fairly easily in the form

$$\gamma_{\mathcal{L},a} = T_h R_{\theta}\rho.$$

This shows that every glide reflection can be written in the above form. What we want, however, is the converse: that every isometry of the above form is a glide reflection. To do this, we need to do the above calculation carefully and figure out precise formulas for h and θ in terms of a, b, and α. We will do this next lecture.

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[1]Note that I use *displacement* rather than *distance* because a could be positive or negative, depending on which direction the glide is. By convention, up will be considered positive and down negative; in the case of a horizontal line, left will be negative and right will be positive.