GROUPS AND SYMMETRY: LECTURE 10

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Last time, we mostly proved a classification of the symmetries of the square. More precisely, we tried to prove the following (see the previous lecture notes for more details on notation):

Proposition 1. *There are precisely eight symmetries of the square* $\{\pm 1 \pm i\}$*, given by*

$$\mathcal{G}_{\{\pm 1 \pm i\}} = \left\{ R_{\pi/2}^k \rho^j : k, j \in \mathbb{Z} \right\}.$$

We realized that it suffices to prove

Proposition 2. *If* $\gamma \in \mathcal{G}_{\{\pm 1 \pm i\}}$*, then* $\gamma(0) = 0$*.*

(Try to deduce Proposition 1 from Proposition 2 without looking at the previous lecture notes!)

Here was our strategy for proving Proposition 2. Let $\mathcal{L}_+$ denote the line segment connecting $-1-i$ to $1+i$, and let $\mathcal{L}_-$ denote the line segment connecting $-1+i$ to $1-i$. (In other words, $\mathcal{L}_\pm$ are the diagonals of the square.) Since $0 \in \mathcal{L}_+ \cap \mathcal{L}_-$, we see that

$$\gamma(0) \in \gamma(\mathcal{L}_+ \cap \mathcal{L}_-) = \gamma(\mathcal{L}_+) \cap \gamma(\mathcal{L}_-).$$

An easy argument will show that $\gamma(\mathcal{L}_+)$ is one of $\mathcal{L}_+$ or $\mathcal{L}_-$, and that $\gamma(\mathcal{L}_-)$ is the other. Therefore,

$$\gamma(0) \in \gamma(\mathcal{L}_+) \cap \gamma(\mathcal{L}_-) = \mathcal{L}_+ \cap \mathcal{L}_- = \{0\}.$$

This immediately implies that $\gamma(0) = 0$.

In the above outline, some of the steps are conjectures which need to be proved. In particular, we need to prove the following:

I. $\gamma(\mathcal{L}_+ \cap \mathcal{L}_-) = \gamma(\mathcal{L}_+) \cap \gamma(\mathcal{L}_-)$
II. $\gamma(\mathcal{L}_+) \cap \gamma(\mathcal{L}_-) = \mathcal{L}_+ \cap \mathcal{L}_-$
III. $\mathcal{L}_+ \cap \mathcal{L}_- = \{0\}$

Claim I follows immediately from a more general statement:

Lemma 3. *For any sets* $A, B \subseteq \mathbb{R}^2$ *and any* $\phi \in \mathcal{G}$*, we have* $\phi(A \cap B) = \phi(A) \cap \phi(B)$*.*

Proof. As usual, we prove this in two steps. We first prove that $\phi(A \cap B) \subseteq \phi(A) \cap \phi(B)$:

$$\begin{aligned} y \in \phi(A \cap B) &\implies \exists x \in A \cap B \text{ such that } y = \phi(x) \\ &\implies \exists x \text{ such that } x \in A, x \in B, \text{ and } y = \phi(x) \\ &\implies y \in \phi(A) \text{ and } y \in \phi(B) \\ &\implies y \in \phi(A) \cap \phi(B). \end{aligned}$$

Next, we prove the reverse inclusion. As we discussed in class, we cannot simply reverse the directions of the arrows (why not?). Here's the proof we came up with:

$$\begin{aligned} y \in \phi(A) \cap \phi(B) &\implies y \in \phi(A) \text{ and } y \in \phi(B) \\ &\implies \exists x_1 \in A, x_2 \in B, \text{ such that } \phi(x_1) = y = \phi(x_2). \end{aligned}$$

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We know every isometry is bijective. In particular, ϕ is injective, so $x_1 = x_2$. We deduce that $x_1 \in A \cap B$, whence

$$y = \phi(x_1) \in \phi(A \cap B).$$ □

This proves claim I from above. We will deduce claim II from the following general result (see the previous lecture notes for a proof):

Lemma 4. *Let $\overline{XY}$ denote the line segment between $X, Y \in \mathbb{C}$. Then for any $\phi \in \mathcal{G}$ and $X, Y \in \mathbb{C}$,*

$$\phi(\overline{XY}) = \overline{\phi(X)\phi(Y)}.$$

We can now prove claim II. Given $\gamma \in \mathcal{G}_{\{\pm 1 \pm i\}}$, where does it send $\mathcal{L}_+$? Lemma 4 shows that it suffices to see where γ sends the endpoints. Both endpoints of $\mathcal{L}_+$ get sent to $\{\pm 1 \pm i\}$. Because γ is injective, they get sent to two different points. Moreover, the distance between the two endpoints of $\mathcal{L}_+$ is $2\sqrt{2}$, so they must get sent either to the endpoints of $\mathcal{L}_+$ or to the endpoints of $\mathcal{L}_-$; in other words, $\gamma(-1-i) = -\gamma(1+i)$. Lemma 4 immediately implies that $\gamma(\mathcal{L}_+) = \mathcal{L}_+$ or $\mathcal{L}_-$.

The same argument shows that $\gamma(-1+i) = -\gamma(1-i)$, so that $\mathcal{L}_-$ gets mapped to either $\mathcal{L}_+$ or $\mathcal{L}_-$. Moreover, by injectivity, $\gamma(\mathcal{L}_+) \neq \gamma(\mathcal{L}_-)$. It follows that one of $\gamma(\mathcal{L}_\pm)$ is $\mathcal{L}_+$, and the other is $\mathcal{L}_-$. Thus,

$$\gamma(\mathcal{L}_+) \cap \gamma(\mathcal{L}_-) = \mathcal{L}_+ \cap \mathcal{L}_-$$

as claimed in II.

Finally, we prove claim III, i.e. that $\mathcal{L}_+ \cap \mathcal{L}_- = \{0\}$. Using our parametrization of lines (as in the proof of Lemma 4 from last lecture) we see that

$$\begin{aligned}\mathcal{L}_+ &= \{(1-t)(1+i) + t(-1-i) : 0 \le t \le 1\} \\ &= \{(1-2t)(1+i) : 0 \le t \le 1\}.\end{aligned}$$

Similarly,

$$\mathcal{L}_- = \{(1-2t)(1-i) : 0 \le t \le 1\}.$$

Thus, if $z \in \mathcal{L}_+ \cap \mathcal{L}_-$, then there exist $s, t \in [0,1]$ such that

$$(1-2t)(1+i) = z = (1-2s)(1-i).$$

This implies

$$1 - 2s = (1-2t)\frac{1+i}{1-i} = (1-2t)i.$$

The left hand side is real and the right hand side is imaginary, which is only possible if both are zero. This implies $s = t = \frac{1}{2}$, whence $z = 0$. This concludes the proof of claim III, and thus, of Proposition 2. □

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