GROUPS AND SYMMETRY: LECTURE 22

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We've been discussing isomorphisms for a bit. Why are we so interested in them? It turns out that many complicated, hard-to-imagine group are isomorphic to very simple or down-to-earth groups. For example, recall the Klein V group

$$V = \{e, a, b, ab\},$$

where $a^2 = e = b^2$ and $ab = ba$. V is a bit strange; among other things, it has four different solutions to $x^2 = e$ (usually, a quadratic equation has at most two solutions). The following proposition shows that there's a much simpler way to think about V.

Proposition 1. *$V \simeq C_2 \times C_2$, where C_2 denotes the cyclic group of order 2.*

Proof. Our first task is to guess an isomorphism $\varphi : V \to C_2 \times C_2$. It will be helpful to label the elements of C_2; let's say it's

$$C_2 := \{e, c\}$$

where $c^2 = e$. (Note that the e here is the identity of C_2, which is different from the identity of V!) Eric suggested defining φ as follows:

$$\varphi(e) = (e, e) \qquad \varphi(a) = (c, e) \qquad \varphi(b) = (e, c) \qquad \varphi(ab) = (c, c)$$

It is now easy to verify that φ is a bijection and a homomorphism. □

Another way to view the above result is:

$$V/C_2 \xrightarrow{\sim} C_2.$$

This isomorphism isn't so surprising. Sometimes, however, it's harder to see how to interpret a quotient. For example, what can you say about $\mathbb{C}^\times/\mathbb{R}_{>0}$? Ray conjectured that this should be isomorphic to $\{R_\theta : \theta \in [0, 2\pi)\}$. His intuition: $\mathbb{C}^\times$ is like the entire complex plane with a hole in the middle, while $\mathbb{R}_{>0}$ is like the positive part of the x-axis. Swiveling this around the circle, we tile all of $\mathbb{C}^\times$ by disjoint copies of $\mathbb{R}_{>0}$.

This type of geometric reasoning is beautiful, but is not always available to us with abstract groups. To help us understand quotient groups in a more unified way, we will prove the following fundamental result.

Theorem 2 (1st Isomorphism Theorem)**.** *Given a homomorphism $\varphi : \Gamma \to H$. Then*

- im $\varphi \leq H$
- ker $\varphi \trianglelefteq \Gamma$
- $\Gamma/\ker \varphi \simeq \operatorname{im} \varphi$

Date: November 25, 2013.

Before discussing the proof and applications, we reviewed some of the terms appearing in the statement of the theorem. First, im φ is the *image* of φ; it's the set of all possible outputs of φ. In symbols:

$$\text{im}\,\varphi := \{\varphi(g) : g \in \Gamma\}.$$

Next, ker φ is the *kernel* of φ; it's the set of all elements of Γ which get killed (i.e. taken to the identity) by φ. Formally:

$$\ker\varphi := \{g \in \Gamma : \varphi(g) = e\}.$$

Thus, the theorem asserts that quotienting out by all the elements which get killed, leaves you with the image.

We'll prove this theorem next time; for now, we apply it to the example from above. Consider the map $\varphi : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$ defined by

$$\varphi(x) := \frac{|x|}{x}.$$

We can write any $x \in \mathbb{C}^\times$ in polar form: $x = re^{i\theta}$. Then

$$\varphi(x) = \frac{r}{re^{i\theta}} = e^{-i\theta},$$

which shows that im $\varphi = \{e^{-i\theta} : \theta \in \mathbb{R}\}$. In other words, the image of φ is simply the unit circle:

$$\text{im}\,\varphi = \{z \in \mathbb{C} : |z| = 1\}.$$

Next, we evaluate the kernel. We have $z \in \ker\varphi$ iff

$$\frac{|z|}{z} = 1,$$

which in turn occurs iff $|z| = z$. But this happens iff $z \in \mathbb{R}_{>0}$. Thus,

$$\ker\varphi = \mathbb{R}_{>0}.$$

Applying the 1st isomorphism theorem, we deduce that $\mathbb{C}^\times/\mathbb{R}_{>0}$ is isomorphic to the unit circle, which is particularly easy to imagine. Incidentally, this agrees with Ray's conjecture, since

$$f : \{R_\theta : \theta \in [0, 2\pi)\} \xrightarrow{\sim} \{z \in \mathbb{C} : |z| = 1\}$$
$$R_\theta \longmapsto e^{i\theta}$$

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