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MATC01: GROUPS AND SYMMETRY

Problem Set 4 – due Friday, October 11th

INSTRUCTIONS:
To receive credit, you must turn this in during the first 5 minutes of lecture on the due date. Please print and attach this page as the first page of your submitted problem set.

PROBLEM	**MARK**
Staple	
Coverpage	
4.1	
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Total	

Please read the following statement and sign below:

I understand that I am not allowed to use the internet to assist (in any way) with this assignment. I also understand that I must write down the final version of my assignment in isolation from any other person.

SIGNATURE:______________________________

Problem Set 4

4.1 (a) Suppose $\gamma \in \mathcal{G}$ satisfies

$$\gamma(2-3i) = 1+2i \qquad \text{and} \qquad \gamma(8+9i) = 13-4i.$$

What is $\gamma(6+5i)$? Justify your answer.

(b) Suppose $\phi \in \mathcal{G}$ satisfies

$$\phi(1+i) = 2-3i \qquad \text{and} \qquad \phi(0) = 3-4i.$$

What is $\phi(-1-i)$? Justify your answer.

4.2 Given a finite set $S \subseteq \mathbb{C}$, define the *centre of mass* of S to be

$$\mu_S := \frac{1}{|S|} \sum_{x \in S} x.$$

(Here $|S|$ denotes the number of elements of S.) For example,

$$A = \{2, 1+i, -1+2i\} \Longrightarrow \mu_A = \frac{1}{3}\Big(2 + (1+i) + (-1+2i)\Big) = \frac{2}{3} + i.$$

The goal of this exercise is to prove that every symmetry of a set fixes the centre of mass of that set.

(a) Given $\gamma \in \mathcal{G}$ and a finite set $S \subseteq \mathbb{C}$. Prove that γ maps the centre of mass of S to the centre of mass of $\gamma(S)$, i.e.

$$\gamma(\mu_S) = \mu_{\gamma(S)}.$$

Here, as usual, $\gamma(S) := \{\gamma(x) : x \in S\}$.

(b) Suppose $S \subseteq \mathbb{C}$ is a finite set and $\gamma \in \mathcal{G}_S$. Prove that $\gamma(\mu_S) = \mu_S$.

4.3 The goal of this exercise is to prove that an isometry is determined by what it does to any triangle. (See part (d) for a more precise statement.)

(a) Given three distinct points $x_1, x_2, x_3 \in \mathbb{C}$ and ψ an orientation-preserving isometry such that $\psi(x_\ell) = x_\ell$ for all $\ell \in \{1, 2, 3\}$. Prove that $\psi = \mathbb{1}$.

(b) Suppose $x, y, z \in \mathbb{C}$ are all distinct, and satisfy

$$\frac{z-y}{y-x} \in \mathbb{R}.$$

Prove that x, y, z are collinear (i.e. that all three points lie on a single line).

(c) Given three distinct points $x_1, x_2, x_3 \in \mathbb{C}$ and ψ an orientation-reversing isometry such that $\psi(x_\ell) = x_\ell$ for all $\ell \in \{1, 2, 3\}$. Prove that x_1, x_2, x_3 must be collinear.

(d) Fix any three non-collinear points $x_1, x_2, x_3 \in \mathbb{C}$. Suppose ϕ and γ are isometries which agree on these three points, i.e. $\phi(x_\ell) = \gamma(x_\ell)$ for all $\ell \in \{1, 2, 3\}$. Prove that $\phi = \gamma$.

4.4 Determine all symmetries of the black-and-white drawing on the course homepage (click on the 'news' tab). Assume the drawing extends infinitely in all directions. You may choose where in the drawing to put the origin (although bear in mind that some choices will make your task easier than others).