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MATC01: GROUPS AND SYMMETRY

Problem Set 8 – due Monday, November 11th

INSTRUCTIONS:
To receive credit, you must turn this in during the first 5 minutes of lecture on the due date. Please print and attach this page as the first page of your submitted problem set.

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Please read the following statement and sign below:

I understand that I am not allowed to use the internet to assist (in any way) with this assignment. I also understand that I must write down the final version of my assignment in isolation from any other person.

SIGNATURE:______________________________

Problem Set 8

8.1 Suppose Γ is a group and H is a subgroup of Γ. Prove that H and Γ have the same identity.

8.2 Given a group Γ and two subgroups X and Y of Γ, define

$$XY := \{xy \,:\, x \in X \text{ and } y \in Y\}.$$

(a) Suppose that H and K are subgroups of a group Γ. Prove that $HK \leq \Gamma$ if and only if $HK = KH$.
[*Hint: You may find it useful to first prove that for any group* Γ, $\Gamma = \{g^{-1} : g \in \Gamma\}$.]

(b) Find an example of a group Γ and two subgroups $H, K \leq \Gamma$ such that HK is *not* a subgroup of Γ.

8.3 Recall that given elements $g_1, g_2, \ldots g_n$ of a group Γ, we defined $\langle g_1, g_2, \ldots, g_n\rangle$ to be the smallest subgroup of Γ which contains all of the g_i's. The goal of this exercise is to make precise what 'smallest' means.

(a) Given a group Γ and two subgroups H and K of Γ, prove that $H \cap K \leq \Gamma$.

(b) Given a group Γ and elements $g_1, g_2, \ldots, g_n \in \Gamma$, formulate a precise definition of $\langle g_1, g_2, \ldots, g_n\rangle$. Your definition must not rely on ambiguous words such as 'smallest', and yet should capture your intuition for what smallest means in this context.

8.4 Prove that any subgroup of a cyclic group must be cyclic. [Recall that a group Γ is *cyclic* iff it can be generated by a single element, ie if there exists $g \in \Gamma$ such that $\Gamma = \langle g\rangle$. For example, $\mathbb{Z}$ is cyclic, since $\mathbb{Z} = \langle 1\rangle$.]

8.5 Let

$$\mathbf{I} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \qquad \text{and} \qquad \mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and set $\mathbf{K} := \mathbf{IJ}$ (matrix multiplication). How many distinct elements are in $\langle \mathbf{I}, \mathbf{J}\rangle$, the subgroup of $GL_2(\mathbb{C})$ generated by $\mathbf{I}$ and $\mathbf{J}$? Write all of them down in terms of $\mathbb{1}$, $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$.

8.6 Prove that $\mathbb{Q}^{\times}$ is not cyclic.