UNIVERSITY OF TORONTO SCARBOROUGH

MATC01 : Groups and Symmetry

FINAL EXAMINATION

December 12, 2013

Duration – 3 hours
Aids: none

NAME (PRINT): KEY
Last/Surname First/Given Name (and nickname)

STUDENT NO: ______________________

Instructions: Please **read** the statement at the bottom of this page and sign below it. This exam consists of 12 pages total. The questions span pages 2 – 10; pages 11 and 12 are scratch paper (you may carefully rip them out if you like). Feel free to do scratchwork on the back of any page, but please try to write your solutions neatly and clearly in the space allotted to each question. More scratch paper is available if you need it. Unless otherwise specified, you may freely refer (without proof) to any results we proved in lecture or on the problem sets. You may also use results stated anywhere on this exam, *even if you aren't able to prove them*. For example, if you can't prove an assertion in part (a) of a problem, you may still use it in part (b).

GOOD LUCK!

Qn. #	Value	Score
1	15	
2	15	
3	20	
4	15	
5	15	
6	20	
Total	100	

TOTAL: ____________

Please read the following statement and sign below:

I understand that any breach of academic integrity is a violation of The Code of Behaviour on Academic Matters. By signing below, I pledge to abide by the Code.

SIGNATURE:______________________

(1) Given a group Γ and a subgroup $H \leq \Gamma$, recall that
$$\Gamma/H := \{[g] : g \in \Gamma\},$$
where $[g] := \{a \in \Gamma : aH = gH\}$.

(a) (5 points) Prove that $[g] = gH$ for any $g \in \Gamma$.

As usual, we prove this in two steps.

$\boxed{[g] \subseteq gH}$
Pick $x \in [g]$. Then by definition, $xH = gH$. Since H is a subgroup of Γ, it contains the identity, whence
$$x \in xH = gH.$$
$\boxed{gH \subseteq [g]}$
Pick $x \in gH$. Then $x = gh$ for some $h \in H$. Since H is a group, we know (from a homework problem) that $hH = H$. It follows by associativity that
$$xH = ghH = gH,$$
whence $x \in [g]$.

(b) (5 points) Without referring to anything from lecture or the homework, prove that for any $a, b \in \Gamma$, either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

Given $a, b \in \Gamma$. If $[a] \cap [b] = \emptyset$, we're done. If $[a] \cap [b] \neq \emptyset$, then there exists $x \in [a] \cap [b]$. This implies that $xH = aH$ and $xH = bH$, whence $aH = bH$. By part (a), we conclude that $[a] = [b]$.

(c) (5 points) Recall that we defined a binary operation on Γ/H by setting
$$[a][b] := [ab].$$
Carefully explain (with a supporting example) why this operation might not be well-defined.

The problem with the 'definition' of the binary operation above is that it's possible for $[a] = [a']$ but $[a][b] \neq [a'][b]$. How? First, observe that $[h] = [e]$ for all $h \in H$. If $[e][g] = [h][g]$ for all $g \in \Gamma$ and all $h \in H$, then (by 'definition') for all $g \in \Gamma$ and $h \in H$ we would have $[g] = [hg]$, i.e. $gH = hgH$, i.e. $g^{-1}hgH = H$. In particular, since $e \in H$, this would imply that $g^{-1}Hg \subseteq H$ for every $g \in \Gamma$. But this may easily fail to be true. For example, consider the group $\mathcal{G}_{\{\pm 1 \pm i\}}$ of symmetries of the square, and the subgroup $H := \{\mathbb{1}, \rho\}$. We have
$$R_{\pi/2}^{-1} \rho R_{\pi/2} = R_\pi \rho \notin H.$$
This tells us that $[\mathbb{1}][R_{\pi/2}] \neq [\rho][R_{\pi/2}]$ in $\mathcal{G}_{\{\pm 1 \pm i\}}/H$, even though $[\mathbb{1}] = [\rho]$.

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(2) (15 points) Prove that $\simeq$ (isomorphism) is an equivalence relation. Do not assume any results from lecture or the homework.

We show that $\simeq$ satisfies the three defining properties of equivalence relations:

(1) *Reflexive.* For any group Γ, we have $\Gamma \simeq \Gamma$, since

$$\begin{aligned} \Gamma &\xrightarrow{\sim} \Gamma \\ g &\longmapsto g \end{aligned}$$

(2) *Symmetry.* Suppose $\Gamma \simeq H$. Then by definition there exists an isomorphism $\phi : \Gamma \xrightarrow{\sim} H$. I claim that $\phi^{-1} : H \xrightarrow{\sim} \Gamma$. It is clear that it is bijective (since ϕ is), so it suffices to prove that ϕ^{-1} is a homomorphism. Given $x, y \in H$, let $a := \phi^{-1}(x)$ and $b := \phi^{-1}(y)$; these are both defined because ϕ is bijective. Then

$$\phi^{-1}(xy) = \phi^{-1}\big(\phi(a)\phi(b)\big) = \phi^{-1}\big(\phi(ab)\big) = ab = \phi^{-1}(x)\phi^{-1}(y),$$

which shows that ϕ^{-1} is a homomorphism. We conclude that ϕ^{-1} is an isomorphism from H to Γ, whence $H \simeq \Gamma$.

(3) *Transitive.* Suppose $\Gamma \simeq H$ and $H \simeq K$. Then there exist isomorphisms

$$\phi : \Gamma \xrightarrow{\sim} H \qquad \text{and} \qquad \psi : H \xrightarrow{\sim} K.$$

I claim that $\psi \circ \phi$ is an isomorphism from Γ to K. It is clearly bijective, since both ψ and ϕ are. Also, it is a homomorphism since for any $a, b \in \Gamma$ we have

$$\psi \circ \phi(ab) = \psi\big(\phi(a)\phi(b)\big) = \psi\big(\phi(a)\big)\psi\big(\phi(b)\big) = \big(\psi \circ \phi(a)\big)\big(\psi \circ \phi(b)\big).$$

We conclude that $\Gamma \simeq K$.

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(3) Recall that $\mathcal{G}$ denotes the group of plane isometries (viewing the plane as $\mathbb{C}$). Consider the set of all symmetries of $\mathbb{Z}$:

$$\mathcal{G}_{\mathbb{Z}} := \{g \in \mathcal{G} : g(\mathbb{Z}) = \mathbb{Z}\}.$$

Thus, for example, T_3 (the translation to the right by 3 units) lives in $\mathcal{G}_{\mathbb{Z}}$, while T_{2i} (the translation up by 2 units) does not.

(a) (10 points) Determine (with proof!) an explicit description of the set $\mathcal{G}_{\mathbb{Z}}$. (In other words, list all the elements of $\mathcal{G}_{\mathbb{Z}}$ and prove that your list is complete.)

I claim that

$$\mathcal{G}_{\mathbb{Z}} = \left\{T_n R_\pi^j \rho^k : n \in \mathbb{Z},\ j, k \in \{0, 1\}\right\}$$

To see this, suppose $\phi \in \mathcal{G}_{\mathbb{Z}}$. From our work on primitive isometries in class, we know that there exist $h \in \mathbb{C}$, $\theta \in [0, 2\pi)$, and $k \in \{0, 1\}$ such that

$$\phi = T_h R_\theta \rho^k.$$

Note that $\phi(0) = h$; since ϕ fixes $\mathbb{Z}$, we deduce that $h \in \mathbb{Z}$. Next, observe that $\phi(1) = h + e^{i\theta}$. Since $h \in \mathbb{Z}$, we see that $e^{i\theta} \in \mathbb{Z}$. But this is only possible if θ is a multiple of π. Thus, ϕ must be of the form $T_n R_\pi^j \rho^k$ for some $n \in \mathbb{Z}$, and $j, k \in \{0, 1\}$. It now remains only to show that every isometry of this form fixes $\mathbb{Z}$.

A quick calculation shows that

$$T_n R_\pi^j \rho^k(m) = n + (-1)^j m$$

for any $m \in \mathbb{Z}$, and it immediately follows that $T_n R_\pi^j \rho^k(\mathbb{Z}) \subseteq \mathbb{Z}$. Moreover, observe that given any $\ell \in \mathbb{Z}$, we have

$$T_n R_\pi^j \rho^k\Big((-1)^j(\ell - n)\Big) = \ell.$$

This shows that $\mathbb{Z} \subseteq T_n R_\pi^j \rho^k(\mathbb{Z})$. Thus, we conclude that $T_n R_\pi^j \rho^k(\mathbb{Z}) = \mathbb{Z}$.

(b) (10 points) Let $T := \{T_n : n \in \mathbb{Z}\}$. Prove that $T \trianglelefteq \mathcal{G}_{\mathbb{Z}}$, and that

$$\mathcal{G}_{\mathbb{Z}}/T \simeq V,$$

where $V = \langle a, b : a^2 = b^2 = e, ab = ba\rangle$ is the Klein V group. [*Hint: Although it's not the only way to prove this, you may find the 1st isomorphism theorem helpful.*]

Consider the map

$$\begin{aligned} \varphi : \quad \mathcal{G}_{\mathbb{Z}} &\longrightarrow V \\ T_n R_\pi^j \rho^k &\longmapsto a^j b^k. \end{aligned}$$

(Note that this is well-defined, since $R_\pi^j = R_\pi^{j'}$ iff $a^j = a^{j'}$, and $\rho^k = \rho^{k'}$ iff $b^k = b^{k'}$.) I claim φ is a homomorphism. To see this, first note that ρ and T_m commute for any integer m. Also, ρ and R_π commute. Finally, $R_\pi^j T_m = T_{(-1)^j m} R_\pi^j$. Applying these relations, we see that

$$\varphi(T_n R_\pi^j \rho^k \circ T_m R_\pi^{j'} \rho^{k'}) = \varphi(T_{n+(-1)^j m} R_\pi^{j+j'} \rho^{k+k'}) = a^{j+j'} b^{k+k'}.$$

Now since V is abelian, we have

$$a^{j+j'} b^{k+k'} = a^j b^k a^{j'} b^{k'} = \varphi(T_n R_\pi^j \rho^k)\varphi(T_m R_\pi^{j'} \rho^{k'}).$$

This proves that φ is a homomorphism.

It is clear that the image of φ is all of V, while the kernel of φ is T. The 1st isomorphism theorem then implies that $T \trianglelefteq \mathcal{G}_{\mathbb{Z}}$ and that

$$\mathcal{G}_{\mathbb{Z}}/T \simeq V.$$

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(4) (15 points) State and prove Cauchy's theorem for abelian groups, using the method discussed in lecture.

This is proved in the lecture notes.

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(5) (15 points) State and prove the 1st isomorphism theorem.

This is proved in the lecture notes.

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(6) Given a group Γ, consider the set $\widehat{\Gamma}$ consisting of all homomorphisms $\chi : \Gamma \to \mathbb{C}^\times$. ($\widehat{\Gamma}$ is called the *dual group* of Γ; the elements χ of $\widehat{\Gamma}$ are called *characters* of Γ.) Given any $\chi, \psi \in \widehat{\Gamma}$, we define the function $\chi\psi$ by setting

$$\chi\psi(g) := \chi(g)\psi(g).$$

Note that the right hand side is ordinary multiplication of complex numbers.

(a) (5 points) Prove that $\widehat{\Gamma}$ is a group under the multiplication defined above. [*Hint: You may assume that multiplication in $\mathbb{C}^\times$ is associative.*]

To prove $\widehat{\Gamma}$ is a group, we check the four group axioms.

(0) *Closure.* Given $\chi, \psi \in \widehat{\Gamma}$, I claim $\chi\psi \in \widehat{\Gamma}$. It suffices to show that $\chi\psi$ is a homomorphism. For any $a, b \in \Gamma$, we have

$$\chi\psi(ab) = \chi(ab)\psi(ab) = \chi(a)\chi(b)\psi(a)\psi(b) = \chi\psi(a)\chi\psi(b)$$

since $\mathbb{C}^\times$ is abelian.

(1) *Associativity.* Given $\chi, \psi, \lambda \in \widehat{\Gamma}$, I claim that $(\chi\psi)\lambda = \chi(\psi\lambda)$. For any $g \in \Gamma$ we have

$$(\chi\psi)\lambda(g) = \big(\chi(g)\psi(g)\big)\lambda(g) = \chi(g)\big(\psi(g)\lambda(g)\big) = \chi(\psi\lambda)(g),$$

by the associativity of $\mathbb{C}^\times$.

(2) *Identity.* The trivial homomorphism $\chi_0 : \Gamma \to \mathbb{C}^\times$ defined by $\chi_0(g) = 1$ for all $g \in \Gamma$ is the identity of $\widehat{\Gamma}$, since it is easily checked that for any $\psi \in \widehat{\Gamma}$ we have $\chi_0\psi = \psi = \psi\chi_0$.

(3) *Inverses.* Given $\chi \in \widehat{\Gamma}$, define the function $\overline{\chi} : \Gamma \to \mathbb{C}^\times$ by

$$\overline{\chi}(g) := \frac{1}{\chi(g)}.$$

This function is well-defined, since $\chi(g) \neq 0$ for all $g \in \Gamma$. It also lives in $\widehat{\Gamma}$, since

$$\overline{\chi}(ab) = \frac{1}{\chi(ab)} = \frac{1}{\chi(a)} \cdot \frac{1}{\chi(b)} = \overline{\chi}(a)\overline{\chi}(b).$$

Finally, observe that $\chi\overline{\chi} = \chi_0$. Thus, we've shown that every element $\chi \in \widehat{\Gamma}$ has an inverse $\overline{\chi} \in \widehat{\Gamma}$.

(b) (5 points) Prove that if Γ is finite, then $|\chi(g)| = 1$ for all $g \in \Gamma$ and all $\chi \in \widehat{\Gamma}$. (Here $|\chi(g)|$ denotes absolute value of the complex number $\chi(g)$, *not* the order of an element.)

By one of the consequences of Lagrange's theorem (sometimes called Euler's theorem), we know that $a^{|\Gamma|} = e$ for every $a \in \Gamma$. It follows that, for any $\chi \in \widehat{\Gamma}$ and any $g \in \Gamma$, we have

$$\chi(g)^{|\Gamma|} = \chi\big(g^{|\Gamma|}\big) = \chi(e) = 1.$$

In particular, we see that

$$|\chi(g)|^{|\Gamma|} = 1,$$

whence $|\chi(g)| = 1$ as claimed.

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(c) (5 points) Let $V = \{e, a, b, ab\}$ denote the Klein V group. Prove that $\widehat{V} \simeq V$.

First, observe that if $\chi \in \widehat{V}$ and $g \in V$, then $\chi(g)^2 = \chi(g^2) = \chi(e) = 1$, whence

$$\chi(g) = \pm 1.$$

Since $\chi(e) = 1$ and $\chi(ab) = \chi(a)\chi(b)$, we see that χ is completely determined by its values at a and b. It follows that $\widehat{V}$ contains precisely four characters:

- χ_0, the trivial character which maps every element of V to 1;
- ψ, which sends a to 1 and b to -1;
- λ, which sends a to -1 and b to 1; and
- $\psi\lambda$, which sends both a and b to -1.

From class, we know that there are precisely two groups of order 4 (up to isomorphism). Since $\chi^2 = \chi_0$ for any $\chi \in \widehat{V}$, we conclude that $\widehat{V}$ isn't cyclic; therefore, it must be isomorphic to V as claimed.

(d) (5 points) Suppose Γ is an arbitrary finite group. Prove that $\widehat{\widehat{\Gamma}}$ (i.e. the dual of the dual) is isomorphic to Γ. You may assume (without proof) that given any non-identity element $g \in \Gamma$, there exists $\chi \in \widehat{\Gamma}$ such that $\chi(g) \neq 1$.

This is left as a challenge problem!

Total Marks = 100 points