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MATC15: NUMBER THEORY

Problem Set 2 – due Thursday, February 14th, during the first five minutes of lecture.

INSTRUCTIONS: Please print and attach this page as the first page of your submitted problem set.

PROBLEM	**MARK**
2.1	
2.2	
2.3	
2.4	
2.5	
2.6	
2.7	
Total	

Please read the following statement and sign below:

I understand that I am not allowed to use the internet to assist (in any way) with this assignment. I also understand that I must write down the final version of my assignment in isolation from any other person.

SIGNATURE:____

Problem Set 2

2.1 Prove that $n^2 - 1$ is prime for exactly one value of n.

2.2 In lecture we discussed Euclid's proof that there are infinitely many primes. The purpose of this problem is to give a different proof, which was apparently discovered only a few years ago by F. Saidak.

(a) Let $n > 1$ be an integer. Show that n and $n+1$ are relatively prime (i.e. have no common factor larger than 1).

(b) Show that $n(n+1)+1$ is relatively prime to both n and $n+1$.

(c) Construct a number which is relatively prime to n, $n+1$, and $n(n+1)+1$.

(d) Use the above to prove that there are infinitely many prime numbers.

(e) Recall that from the Euclidean proof we were able to deduce that $\pi(x) \geq \log\log x$. What's the best lower bound you can deduce from Saidak's proof?

2.3 In this problem, you will prove the infinitude of primes in yet another way. Consider the sets

$$A_n = \{1 + k(n!) : 1 \leq k \leq n\},$$

where the parameter k is assumed to be an integer.

(a) Show that any two elements of A_n are relatively prime.

(b) Use this to give another proof that there are infinitely many primes.

(c) What's the best lower bound on $\pi(x)$ you can derive from this proof?

2.4 Fix $k \geq 1$. Prove that $\frac{(\log x)^k}{x} \to 0$ as $x \to \infty$. [*Hint: there is a very short proof!*]

2.5 Prove that $\displaystyle\int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}$. (Recall that $f(x) \sim g(x)$ means $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.)

2.6 In this exercise, we fill in some gaps from proofs given in class.

(a) Prove that $\log x/\log 2 + 1 \leq 2\log x$ for all sufficiently large x. What does 'sufficiently large' mean? Be as specific as possible (i.e. give a value x_0 such that for all $x \geq x_0$, the inquality above holds; this requires proof!).

(b) Prove that $\log x/\log 3 + 1 \leq \log x$ for all x large enough. What does 'sufficiently large' mean? As above, be as specific as possible.

(c) Prove that for all sufficiently large n,

$$\frac{(1+\log 2)n}{\log \frac{n}{2}} \leq \frac{2n}{\log n}.$$

What does sufficiently large mean? As above, be as specific as possible.

(d) Prove that for all sufficiently large n,

$$\frac{2n \log 2}{\log(2n+1)} - 1 \geq \frac{\log 2}{2} \cdot \frac{2n}{\log 2n}$$

What does sufficiently large mean? As above, be as specific as possible.

2.7 Let $I_n = \displaystyle\int_0^1 x^n(1-x)^n \, dx$, as in our proof of Chebyshev's lower bound. Prove that $0 < I_n \leq \frac{1}{4^n}$ for all $n \in \mathbb{N}$.