Instructor: Leo Goldmakher

NAME: ____________________

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

Problem Set 4 (due November 15, 2010 at the start of lecture)

INSTRUCTIONS: Please attach this page as the first page of your submitted problem set.

PROBLEM	**MARK**
4.1	
4.2	
4.3	
4.4	
4.5	
4.6	
Total	

Problem Set 4

4.1 In lecture we discussed Euclid's proof that there are infinitely many primes. The purpose of this problem is to give a different proof, which was apparently discovered only a few years ago by Filip Saidak (Amer. Math. Monthly 113 (2006), no. 10, pages 937-938).

(a) Let $n > 1$ be an integer. Show that n and $n + 1$ are relatively prime (i.e. have no common factor larger than 1).

(b) Show that $n(n + 1) + 1$ is relatively prime to both n and $n + 1$.

(c) Construct a number which is relatively prime to n, $n + 1$, and $n(n + 1) + 1$.

(d) Use the above to prove that there are infinitely many prime numbers.

4.2 Suppose ABC is a right triangle, with a right angle at B. Let M be the midpoint of side $\overline{AC}$, and form the line segment $\overline{BM}$. Prove that $\overline{BM} \cong \overline{AM}$.

4.3 Do parts (a)-(c) of problem 3.8 from Eves (page 98).

4.4 (a) Use the Euclidean algorithm to find the greatest common divisor of 423 and 957.

(b) Use the Euclidean algorithm to find the greatest common divisor of 273 and 442.

4.5 Recall that the area of a trapezoid is the height multiplied by the average of the two parallel bases. Use this to do problem 5.3 (c) of Eves (pages 155-156).

4.6 In class we discussed a theorem of Euclid's that any positive integer can be written as a product of prime numbers in an essentially unique way; this is usually called the *Fundamental Theorem of Arithmetic*. A different way of expressing this is to say that for any positive integer n there is a unique sequence of integers $n_p \geq 0$ such that

$$n = 2^{n_2} 3^{n_3} 5^{n_5} 7^{n_7} 11^{n_{11}} \cdots$$

where the product is over all primes. For example, if $n = 12$, then $n_2 = 2$, $n_3 = 1$, and $n_p = 0$ for all primes $p \geq 5$. We will use a similar notation for letters other than n, e.g. if $a = 25$, then $a_5 = 2$ and $a_p = 0$ for all primes $p \neq 5$.

(a) Given positive integers n and d, show that n is a multiple of d if and only if $d_p \leq n_p$ for every prime p.

(b) Carefully justify the steps of the proof below. You may want to use part (a).

<u>Theorem:</u> For any positive integer n, $\sqrt{n}$ is either an integer *or* it is irrational.

<u>Proof:</u>

> Suppose $\sqrt{n}$ is rational. Then we can write $\sqrt{n} = \frac{a}{b}$, from which we deduce that $n = \frac{a^2}{b^2}$. Therefore a^2 is divisible by b^2. But this means that a is divisible by b, which implies that $\sqrt{n}$ is an integer.

QED