THE CANTOR-SCHRÖDER-BERNSTEIN THEOREM

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ABSTRACT. We give a proof of the Cantor-Schröder-Bernstein theorem: if A injects into B and B injects into A, then there is a bijection between A and B. This seemingly obvious statement is surprisingly difficult to prove. The proof presented here is modeled on the argument given in section 2.6 of [1]; the only differences are expository.

1. STATEMENT OF THE THEOREM AND SKETCH OF PROOF

Given two sets X and Y, we will write $X \sim Y$ to denote the existence of a bijection from X to Y. One easily checks that $\sim$ is transitive, i.e. if $X \sim Y$ and $Y \sim Z$, then $X \sim Z$. The purpose of this note is to prove the following result:

Theorem 1 (Cantor-Schröder-Bernstein)**.** *If $f : A \to B$ and $g : B \to A$ are both injections, then $A \sim B$.*

Here's the strategy of the proof. First, we apply f to all of A to obtain a set $B_1 \subseteq B$. Next, apply g to all of B_1 to get a set $A_2 \subseteq A$. Iterating this, we keep bouncing back and forth between smaller and smaller subsets of A and B until the process stabilizes and we end up with some sets $\overline{A} \subseteq A$ and $\overline{B} \subseteq B$ for which $f(\overline{A}) = \overline{B}$ and $g(\overline{B}) = \overline{A}$. This implies that $\overline{A} \sim \overline{B}$. The next task is to show that $A - \overline{A} \sim B - \overline{B}$, which turns out to be not so hard. Finally, we conclude that $A \sim B$.

2. PROOF OF THE THEOREM

Let

$$\begin{array}{cc} A_0 = A & B_0 = B \\ A_1 = g(B_0) & B_1 = f(A_0) \\ A_2 = g(B_1) & B_2 = f(A_1) \\ \vdots & \vdots \\ A_n = g(B_{n-1}) & B_n = f(A_{n-1}) \\ \vdots & \vdots \end{array}$$

We make two observations, both of which are consequences of injectivity:

Lemma 2.

$$A = A_0 \sim B_1 \sim A_2 \sim B_3 \sim A_4 \sim \cdots$$
$$B = B_0 \sim A_1 \sim B_2 \sim A_3 \sim B_4 \sim \cdots$$

Lemma 3.

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$$
$$B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots$$

Remark. *If* $A_N = A_{N+1}$ *for some* N*, then Lemma 2 immediately implies that* $A \sim B$*. We will therefore assume that the inclusions in Lemma 3 are all strict.*

At this point, it may seem that we have all the tools necessary to prove the theorem. Indeed, Lemma 2 implies that $A \sim B_1$ and $A_1 \sim B$, whence we might be tempted to deduce that $A \cup A_1 \sim B \cup B_1$; since $A \supseteq A_1$ and $B \supseteq B_1$, this would immediately imply that $A \sim B$. The problem with this approach is that the two conditions $X_1 \sim Y_1$ and $X_2 \sim Y_2$ do *not* imply $X_1 \cup X_2 \sim Y_1 \cup Y_2$ in general. However, we have the following result.

Lemma 4. *Suppose we have sets* $\{X_i\}$ *and* $\{Y_i\}$ *satisfying* $X_i \sim Y_i$ *for all* i*. If all the* X_i *are pairwise disjoint, and all the* Y_i *are pairwise disjoint, then*

$$\bigcup_i X_i \sim \bigcup_i Y_i.$$

Thus, to continue our line of argument, we require analogues of the sets A_i which are pairwise disjoint. Fortunately, this isn't difficult to cook up. For each n, set $A_n^* := A_n - A_{n+1}$. By the remark directly after Lemma 3, we see that all of the sets A_n^* are nonempty; moreover, they are pairwise disjoint. Similarly defining sets B_n^*, we have the following analogue of Lemma 2:

Lemma 2*.

$$A_0^* \sim B_1^* \sim A_2^* \sim B_3^* \sim A_4^* \sim \cdots$$
$$B_0^* \sim A_1^* \sim B_2^* \sim A_3^* \sim B_4^* \sim \cdots$$

We can now carry out our prior attack. Since $A_0^* \sim B_1^*$ and $A_1^* \sim B_0^*$, Lemma 4 implies that

$$A_0^* \cup A_1^* \sim B_0^* \cup B_1^*.$$

More generally, we deduce that

$$A_{2n}^* \cup A_{2n+1}^* \sim B_{2n}^* \cup B_{2n+1}^*$$

for all n. Taking the union over all n and once again applying Lemma 4, we conclude that

$$\bigcup_{n\geq 0} A_n^* \sim \bigcup_{n\geq 0} B_n^*.$$

At this point, we're nearly finished; the left hand side looks a lot like A, and the right hand side like B. For brevity, denote the left hand side by $\widetilde{A}$ and the right hand side by $\widetilde{B}$, so that we have

$$\widetilde{A} \sim \widetilde{B}. \tag{1}$$

Does $A = \widetilde{A}$? Somewhat surprisingly, the answer is no in general. To fill in the missing piece, we introduce one final notation: set $\overline{A} = \bigcap_{n\geq 0} A_n$ and $\overline{B} = \bigcap_{n\geq 0} B_n$. We have the following result:

Lemma 5. $A = \overline{A} \cup \widetilde{A}$ *is a partition of* A*, and* $B = \overline{B} \cup \widetilde{B}$ *is a partition of* B*.*

Once again by Lemma 4, we see that to conclude the proof of the theorem it suffices to show that $\overline{A} \sim \overline{B}$. This is an immediate consequence of the following:

Lemma 6. $f(\overline{A}) = \overline{B}$ *and* $g(\overline{B}) = \overline{A}$*.*

This concludes the proof of the theorem. ∎

REFERENCES

[1] A. N. Kolmogorov and S. V. Fomin, *Introductory Real Analysis*, translated by R. A. Silverman, Dover Publications, New York, 1975. 1

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