THE EULER CHARACTERISTIC OF A GRAPH

LEO GOLDMAKHER

ABSTRACT. Euler's theorem on the Euler characteristic of planar graphs is a fundamental result, and is usually proved using induction. Here I present a totally different proof, discovered jointly by Stephanie Mathew (an undergraduate at the time) and Red Burton (a computer program). I learned of this proof thanks to the remarkable website of David Eppstein, who has compiled twenty proofs of Euler's formula (as well as many other lovely expository articles).

1. EULERIAN TOURS AND EULER'S FORMULA

Given a planar[1] graph G, let $V(G)$ denote the number of vertices of G, $E(G)$ the number of edges of G, and $F(G)$ the number of faces of G (i.e. the number of two dimensional pieces G partitions the plane into). When there is no ambiguity, we shall simply write V, E, and F. Our goal is to prove the following fundamental discovery of Euler:

Theorem 1.1 (Euler's Formula)**.** *For any connected planar graph, $V - E + F = 2$.*

Remark. The quantity $V - E + F$ is called the *Euler characteristic* of a graph.

The main idea in our proof is to study the Euler characteristic of a particularly nice family of graphs. Recall that a graph has an *Eulerian tour* iff there exists a path that starts and ends at the same vertex of the graph, visiting every vertex of the graph along the way and traversing each edge of the graph precisely once. Euler, inspired by the bridges of Königsberg, famously characterized all such graphs:

Theorem 1.2. *A graph has an Eulerian tour if and only if it's connected and every vertex has even degree.*

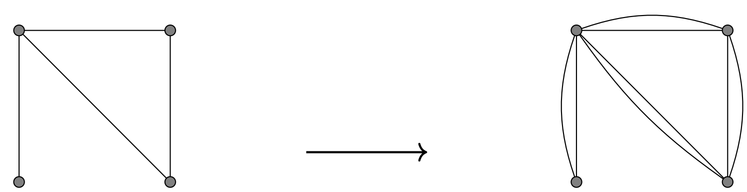
We relegate the proof of this well-known result to the last section.

Our approach to Theorem 1.1 is to reduce it to the following special case:

Proposition 1.3. *Euler's formula $V - E + F = 2$ holds for any graph that has an Eulerian tour.*

With this in hand, the proof of Theorem 1.1 becomes a simple matter. The following argument was devised by Stephanie Mathew when she was a second-year engineering undergraduate at the University of Houston.

Proof of Theorem 1.1. Given a connected planar graph G, we form a new graph G' on the same set of vertices as G as follows. For every edge e in G, add an edge parallel to it (see picture below). Note that each edge we add in this way adds precisely one face, whence the Euler characteristic of G' is the same as the Euler characteristic of G. But by Theorem 1.2 we know G' has an Eulerian tour, so Proposition 1.3 implies that the Euler characteristic of G' is 2. It follows that the Euler characteristic of G is also 2, as claimed. □



The graph G *The graph G'*

[1]Recall that a graph is *planar* iff it can be drawn in the plane in such a way that edges only intersect at shared vertices.

It remains only to prove Proposition 1.3. The proof was inspired by an observation of Red Burton, a computer program written by Siemion Fajtlowicz that specializes in making conjectures. Red Burton's key insight was that each time one revisits a vertex on an Eulerian tour, this adds a face to the graph. Formalizing this quickly leads to the following proof:

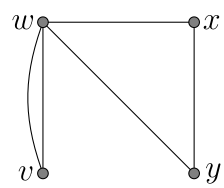
Proof of Proposition 1.3. Let G be a graph that has an Eulerian tour. This Eulerian tour visits every vertex at least once; let $r(v)$ denote the number of times the Eulerian tour revisits v (see example below). Since we are considering an Eulerian tour, we immediately see that

$$\sum_{v \in V} r(v) = E + 1 - V.$$

On the other hand, each visit to a vertex (beyond the first visit) creates a new interior face, which produces a different interpretation of the left hand side of the above equation:

$$\sum_{v \in V} r(v) = F - 1.$$

Combining the above two identities proves the claim. □



$r(x) = r(y) = 0$, *while* $r(v) = r(w) = 1$

2. EULERIAN TOURS

In the above proof we took for granted Theorem 1.2, Euler's characterization of graphs with Eulerian tours. The goal of this section is to prove this result. We handle the two directions of the theorem separately.

Proposition 2.1. *If G has an Eulerian tour, then it is connected and every vertex has even degree.*

Proof. Let's say there's an Eulerian tour that starts and ends at vertex v_0:

$$v_0 \to v_1 \to v_2 \to \cdots \to v_k \to \cdots \to v_{n-1} \to v_0.$$

Here we denote the k^{th} vertex visited along the tour by v_k, but note that the v_i need not be distinct; it's possible for the tour to revisit a vertex. However, since every single vertex of G must appear somewhere on the tour, we immediately deduce that G must be connected. Moreover, because each edge is traversed exactly once along our tour, we see that

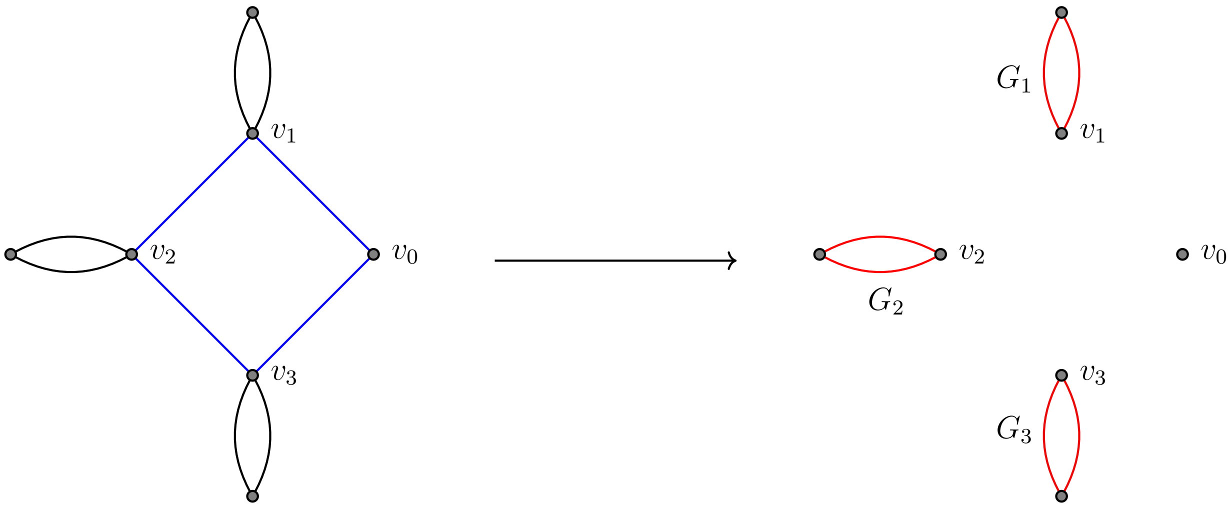
$$\deg(v) = 2 \cdot \#\{i \in [0, n-1] : v_i = v\}.$$

for any vertex v of G. In particular, the degree of each vertex is even. □

Having dispatched with the forward direction of Theorem 1.2, we now prove the converse:

Proposition 2.2. *Any connected graph whose vertices all have even degree has an Eulerian tour.*

We sketch the proof before writing it down formally. Given a graph G, pick a random vertex v_0 and take a random walk along the edges of G, labeling each vertex on your walk and deleting any edge you traverse. Eventually you'll get back to v_0 and get stuck (see illustration below). Whatever's left of the original graph consists of a bunch of small connected graphs: G_1 containing v_1, G_2 containing v_2, etc. All the vertices in all the G_i's still have even degree, so by induction, each of these graphs must have an Eulerian tour. This gives us an Eulerian tour of our original graph G: start at v_0, walk to v_1, take an Eulerian tour of G_1, walk to v_2, take an Eulerian tour of G_2, etc!



The original graph G *What's left of G after our random walk*

Proof. We induct on the number of edges of the graph. Consider a connected graph G in which each vertex has even degree, and pick any vertex v_0 of G. Start walking from v_0 along edges at random, labeling v_i the i^{th} vertex you arrive at along the walk. As you walk, delete every edge you traverse. Keep walking until you get stuck somewhere (i.e. until you get to a vertex where there are no edges you can take to leave). This vertex must be v_0, since otherwise it would have odd degree!

Now consider the graph G' that remains after taking the above walk. First observe that since we walked along an Eulerian tour, we've removed an even number of edges from each of the vertices v_i; this means that every vertex in G' still has even degree. For each $j \geq 1$, let G_j denote the connected component of G' that contains v_j, unless v_j is already a vertex in some previously defined G_i with $i < j$, in which case set G_j to be the lone vertex v_j with no edges.

Note that each G_j is a connected graph in which every vertex has even degree. Since G_j has a smaller number of edges than G, our inductive hypothesis guarantees that G_j has an Eulerian tour for every j. We've thus constructed an Eulerian tour of our original graph G: starting at v_0, walk to v_1 and take the Eulerian tour of G_1, then walk to v_2 and take the Eulerian tour of G_2, etc. $\square$

This completes our proof of Theorem 1.2.

DEPT OF MATHEMATICS & STATISTICS, WILLIAMS COLLEGE, WILLIAMSTOWN, MA, USA
E-mail address: leo.goldmakher@williams.edu