DIFFERENTIATION UNDER THE INTEGRAL SIGN

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In his autobiography *Surely you're joking, Mr. Feynman*, Richard Feynman discusses his "different box of tools". Among other things, he mentions a tool he picked up from the text *Advanced Calculus* by Woods, of differentiating under the integral sign – "it's a certain operation... that's not taught very much in the universities". It seems some things haven't changed in the way calculus is taught, since I was never exposed to this trick in classes either. However, I've accidentally stumbled upon several elegant applications of the maneuver, and thought I'd write a couple of them down.

1. Derivation of the gamma function

An old problem is to extend the factorial function to non-integer arguments. This was resolved by Euler, who discovered two formulas for $n!$ (one an integral, the other an infinite product) which make sense even when n is not an integer. We derive one of Euler's formulas by employing the trick of differentiating under the integral sign. I learned about this method from the website of Noam Elkies, who reports that it was used by Inna Zakharevich in a Math 55a problem set.

Let

$$F(t) = \int_0^\infty e^{-tx}\, dx.$$

The integral is easily evaluated, so that $F(t) = \frac{1}{t}$ for all $t > 0$. Differentiating F with respect to t easily leads to the identity

$$F'(t) = -\int_0^\infty x\, e^{-tx}\, dx = -\frac{1}{t^2}.$$

Taking further derivatives yields

$$\int_0^\infty x^n e^{-tx}\, dx = \frac{n!}{t^{n+1}}$$

which immediately implies the formula

$$n! = \int_0^\infty x^n e^{-x}\, dx.$$

The right hand side is the famous gamma function, and does not depend on n being an integer.

2. A substitute for contour integration

A standard application of complex analysis is to calculate the integral

$$\int_{-\infty}^\infty \frac{\sin x}{x}\, dx.$$

This integral is difficult to handle by standard methods, because the antiderivative of $\frac{\sin x}{x}$ cannot be expressed in terms of elementary functions. We will calculate this integral using two tricks: differentiating under the integral sign, and representing $\sin x$ in terms of complex exponentials.

First, observe that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = 2\int_{0}^{\infty} \frac{\sin x}{x}\, dx$$

so that it suffices to evaluate the integral on the right hand side. Set

$$G(t) = \int_{0}^{\infty} \frac{\sin x}{x} e^{-tx}\, dx.$$

This clearly converges for all $t \geq 0$, and our aim is to evaluate $G(0)$. We have

$$G'(t) = -\int_{0}^{\infty} e^{-tx} \sin x\, dx.$$

This integral can be explicitly evaluated. One approach, which is elegant but somewhat *ad hoc*, is to integrate by parts twice. A different method, which requires more calculation but is vastly more general, is to substitute $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$ and integrate the result. Either way, we find that for every $t > 0$,

$$G'(t) = -\frac{1}{1+t^2}.$$

It follows that $G(t) = C - \arctan t$ for some constant C. Since $G(t)$ tends to 0 as $t \to \infty$ while $\arctan t \to \frac{\pi}{2}$, we see that $C = \frac{\pi}{2}$; in other words,

$$G(t) = \frac{\pi}{2} - \arctan t$$

for all $t > 0$. Letting t tend to 0 from the right, we conclude that $G(0) = \frac{\pi}{2}$. This implies that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi.$$

3. A CURIOUS FORMULA

Let $G(t)$ be as before. By examining $G'(t)$ above we found the identity

$$\int_{0}^{\infty} e^{-tx} \sin x\, dx = \frac{1}{1+t^2}.$$

Differentiating both sides further with respect to t yields the following identity:

$$\int_{0}^{\infty} x^n e^{-tx} \sin x \frac{dx}{x} = \frac{(n-1)!}{(1+t^2)^n}\, g_n(t) \tag{1}$$

where

$$g_n(t) = \sum_{k\geq 0} (-1)^k \binom{n}{2k+1} t^{n-(2k+1)}.$$

Observe* that $g_n(t) = \frac{i}{2}\Big((t-i)^n - (t+i)^n\Big)$.

*I'm grateful to B. Sury for pointing this out to me.

Substituting this into (1) and tidying up a bit leads to the following curious result, which I have not seen before.

Theorem 1. *For any real number $t \geq 0$ and any integer $n \geq 1$ we have*

$$\int_0^\infty x^n e^{-tx} \sin x \frac{dx}{x} = \frac{\sin n\theta}{(1+t^2)^{n/2}}(n-1)!$$

where $\theta := \arcsin \frac{1}{\sqrt{1+t^2}}$.

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