Math Faculty Seminar Series

# On the Duffin-Schaeffer conjecture


Dimitris Koukoulopoulos, Class of 1960s Speaker

Département de mathématiques et de statistique, Université de Montréal


*Monday, October 21st, 1pm in Stetson Court Classroom 110*


**Abstract.** Given any real number $\alpha$, Dirichlet proved that there are infinitely many reduced fractions $a/q$ such that $|\alpha - a/q| \leq q^{-2}$. Can we get closer to $\alpha$ than that? For certain "quadratic irrationals" such as $\alpha = \sqrt{2}$ the answer is no. However, Khinchin proved that if we exclude such thin sets of numbers, then we can do much better. More precisely, let $(\Delta_q)_{q=1}^{\infty}$ be a sequence of error terms such that $q^2\Delta_q$ decreases. Khinchin showed that if the series $\sum_{q=1}^{\infty} q\Delta_q$ diverges, then almost all $\alpha$ (in the Lebesgue sense) admit infinitely many reduced rational approximations $a/q$ such that $|\alpha - a/q| \leq \Delta_q$. Conversely, if the series $\sum_{q=1}^{\infty} q\Delta_q$ converges, then almost no real number is well-approximable with the above constraints. In 1941, Duffin and Schaeffer set out to understand what is the most general Khinchin-type theorem that is true, i.e., what happens if we remove the assumption that $q^2\Delta_q$ decreases. In particular, they were interested in choosing sequences $(\Delta_q)_{q=1}^{\infty}$ supported on sparse sets of integers. They came up with a general and simple criterion for the solubility of the inequality $|\alpha - a/q| \leq \Delta_q$. In this talk, I will explain the conjecture of Duffin-Schaeffer as well as the key ideas in recent joint work with James Maynard that settles it.