EULER'S NUMBER e IS IRRATIONAL

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ABSTRACT. I give two quick proofs that e is irrational. The first (using Taylor series) is folklore; the second (using isolated points) was shown to me by Trevor Wooley.

The goal of this note is to give two quick proofs of the following result:

Theorem. $e \notin \mathbb{Q}$.

PROOF 1: TAYLOR SERIES.

Suppose $e \in \mathbb{Q}$, say, $e = \frac{a}{N}$ for some whole numbers a and N. From the Taylor expansion (around 0) of e^x, we have

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(N-1)!} + \frac{1}{N!} + \frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots$$

It follows that

$$e \cdot N! = \underbrace{N! + N! + \frac{N!}{2!} + \cdots + N + 1}_{\in \mathbb{Z}} + \underbrace{\frac{1}{N+1} + \frac{1}{(N+1)(N+2)} + \cdots}_{\in (0,1)}$$

Since the left hand side is an integer while the right hand side is not, we have reached a contradiction. □

PROOF 2: ISOLATED POINTS.

Recall that an *isolated point* of a set is a point which is far away from any other point in the set. More precisely, given a set $S \subseteq \mathbb{R}$, we say p is *isolated* in S iff there exists a nonempty open interval I such that $I \cap S = \{p\}$. Our proof of the irrationality of e will hinge on the following nice property of rationals.

Exercise 1. *Prove: If $\alpha \in \mathbb{Q}$, then 0 is an isolated point in $\mathbb{Z} + \mathbb{Z}\alpha$. (Here $\mathbb{Z} + \mathbb{Z}\alpha := \{x + y\alpha : x, y \in \mathbb{Z}\}$.)*

Now consider the function $F(n) := \displaystyle\int_0^1 x^n e^x \, dx$.

Exercise 2. *Let $F(\mathbb{N}) := \{F(n) : n \in \mathbb{N}\}$. Prove that $F(\mathbb{N}) \subseteq \mathbb{Z} + \mathbb{Z}e$.*

Exercise 3. *Prove that 0 isn't isolated inside $\mathbb{Z} + \mathbb{Z}e$.* [Hint: Consider $\lim_{n\to\infty} F(n)$.]

Combining this with Exercise 1 yields the theorem. □

Remark. Exercise 1 exhibits a property all rational numbers enjoy but which doesn't hold for e. In fact, this property fails to hold for every irrational, and therefore can be viewed as a characterization of rational numbers.

Exercise 4. *Prove that $\alpha \in \mathbb{Q}$* if and only if *0 is an isolated point in $\mathbb{Z} + \mathbb{Z}\alpha$.*

Question. *Can these proofs be adapted to show that $e^2 \notin \mathbb{Q}$? or $e^3 \notin \mathbb{Q}$? I'd love to hear your thoughts!*

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Date: June 9, 2015.