PALEY'S THEOREM, REVISITED

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1. Completion

In number theory, one often deals with *incomplete* sums, i.e. sums over an unnaturally short range. One trick which has proved useful is to complete the sum. We will discuss an illustrative case of this technique.

1.1. **Preliminary example.**

Suppose one wishes to bound the incomplete sum

$$\sum_{n\leq N} \frac{\overline{\chi}(n)}{n} e(n\alpha)$$

where χ (mod q) is primitive, $N \leq q$, and $\alpha \in [0,1)$. We have

$$\sum_{n\leq N} \frac{\overline{\chi}(n)}{n} e(n\alpha) = \sum_{n\leq q} \frac{\overline{\chi}(n)}{n} e(n\alpha)\, \delta_{[1,N]}(n)$$

where δ_S is the characteristic function of the set S. For all $n \in \mathbb{Z}$, we have

$$\delta_{[1,N]}(n) = \sum_{a\leq N} \delta_a(n)$$

where $\delta_a = \delta_{\{a\}}$. The trick is now to realize δ_a in terms of some explicit functions. There are many ways to do this; for the purposes of this example, a convenient choice is

$$\delta_a(n) = \int_0^1 e(a\theta)e(-n\theta)\, d\theta.$$

(Note that this is valid only when $a, n \in \mathbb{Z}$!) We find

$$\begin{aligned}\sum_{n\leq N} \frac{\overline{\chi}(n)}{n} e(n\alpha) &= \sum_{n\leq q} \frac{\overline{\chi}(n)}{n} e(n\alpha) \sum_{a\leq N} \int_0^1 e(a\theta)e(-n\theta)\, d\theta \\ &= \int_0^1 \left(\sum_{n\leq q} \frac{\overline{\chi}(n)}{n} e\big(n(\alpha-\theta)\big)\right) \left(\sum_{a\leq N} e(a\theta)\right) d\theta.\end{aligned}$$

In particular, we have

$$\left|\sum_{n\leq N} \frac{\overline{\chi}(n)}{n} e(n\alpha)\right| \leq \max_{\theta\in[0,1]} \left|\sum_{n\leq q} \frac{\overline{\chi}(n)}{n} e(n\theta)\right| \cdot \int_0^1 \left|\sum_{a\leq N} e(a\theta)\right| d\theta.$$

Thus we have a bound on the incomplete sum in terms of a complete sum multiplied by a factor we hope is small. How small is it? The trivial bound gives

$$\int_0^1 \left|\sum_{a\leq N} e(a\theta)\right| d\theta \leq N$$

If one instead applies Cauchy-Schwarz, this bound can be significantly improved:

$$\begin{aligned}\int_0^1 \left|\sum_{a\le N} e(a\theta)\right| d\theta &\le \left(\int_0^1 \left|\sum_{a\le N} e(a\theta)\right|^2 d\theta\right)^{1/2}\\ &= \left(\sum_{a_1,a_2\le N} \int_0^1 e\Big((a_1-a_2)\theta\Big)\, d\theta\right)^{1/2}\\ &= \left(\sum_{a_1,a_2\le N} \delta_{a_1}(a_2)\right)^{1/2}\\ &= \sqrt{N}\end{aligned}$$

Finally, if one is much more careful, it is possible (by summing the geometric series, splitting the integral into intervals of length $1/N$, and playing around with geometry) to obtain an asymptotic for the integral:

$$\int_0^1 \left|\sum_{a\le N} e(a\theta)\right| d\theta = \frac{4}{\pi^2}\log N + O(1).$$

This yields

$$\frac{1}{\log N}\left|\sum_{n\le N} \frac{\overline{\chi}(n)}{n} e(n\alpha)\right| \le \left(\frac{4}{\pi^2} + o(1)\right) \max_{\theta\in[0,1]} \left|\sum_{n\le q} \frac{\overline{\chi}(n)}{n} e(n\theta)\right|.$$

1.2. **A slight modification.** For our intended application, we'll need to work with a slightly different example:

$$\sum_{1\le|n|\le N} \frac{\overline{\chi}(n)}{n} e(n\alpha)$$

Running through the same argument as above, we find

$$\begin{aligned}\sum_{1\le|n|\le N} \frac{\overline{\chi}(n)}{n} e(n\alpha) &= \sum_{1\le|n|\le q} \frac{\overline{\chi}(n)}{n} e(n\alpha)\, \delta_{[-N,N]}(n)\\ &= \int_0^1 \left(\sum_{1\le|n|\le q} \frac{\overline{\chi}(n)}{n} e\big(n(\alpha-\theta)\big)\right) D_N(\theta)\, d\theta\end{aligned}$$

where $D_N(\theta)$ is the *Dirichlet kernel*:

$$D_N(\theta) = \sum_{|n|\le N} e(n\theta).$$

Note that $D_N(\theta)$ is always real-valued and satisfies $\int_0^1 D_N(\theta)\, d\theta = 1$. This looks promising, but unfortunately it is not quite this integral we need to be small:

$$\left|\sum_{1\le|n|\le N} \frac{\overline{\chi}(n)}{n} e(n\alpha)\right| \le \max_{\theta\in[0,1]} \left|\sum_{1\le|n|\le N} \frac{\overline{\chi}(n)}{n} e(n\theta)\right| \cdot \int_0^1 |D_N(\theta)|\, d\theta$$

It is a good exercise to determine the size of $\int_0^1 |D_N(\theta)|\, d\theta$.

1.3. **Smoothing.** To further improve this method, we introduce an auxiliary technique called *smoothing*. We illustrate how this is done using the same example as above.

Suppose $\phi(x)$ is a nice, smooth function, to be explicitly chosen later. Using the same approach (and the same representation of δ_a) as above, we find

$$\sum_{1\le|n|\le N} \frac{\overline{\chi}(n)}{n} e(n\alpha)\phi(n) = \sum_{1\le|n|\le q} \frac{\overline{\chi}(n)}{n} e(n\alpha) \sum_{|a|\le N} \phi(a) \int_0^1 e(a\theta)e(-n\theta)\, d\theta$$
$$= \int_0^1 \left(\sum_{1\le|n|\le q} \frac{\overline{\chi}(n)}{n} e\Big(n(\alpha-\theta)\Big) \right) \Phi_N(\theta)\, d\theta$$

where

$$\Phi_N(\theta) = \sum_{|a|\le N} \phi(a)e(a\theta).$$

It follows that

$$\left| \sum_{1\le|n|\le N} \frac{\overline{\chi}(n)}{n} e(n\alpha)\phi(n) \right| \le \max_{\theta\in[0,1]} \left| \sum_{1\le|n|\le q} \frac{\overline{\chi}(n)}{n} e(n\theta) \right| \cdot \int_0^1 |\Phi_N(\theta)|\; d\theta.$$

What do we win by this? Well, if we could choose $\phi(n)$ so that on one hand,

$$\sum_{1\le|n|\le N} \frac{\overline{\chi}(n)}{n} e(n\alpha)\phi(n) \approx \sum_{1\le|n|\le N} \frac{\overline{\chi}(n)}{n} e(n\alpha),$$

while on the other hand $|\Phi_N(\theta)|$ has small mass, we would have a strong bound. Fortunately for us, Fejér cleverly constructed precisely such a function, the *Fejér kernel*, defined

$$\Phi_N(\theta) = \sum_{|a|\le N} \left(1 - \frac{|a|}{N}\right) e(a\theta).$$

It can be shown[1] that this is real, non-negative, and has unit mass:

$$\int_0^1 |\Phi_N(\theta)|\; d\theta = \int_0^1 \Phi_N(\theta)\, d\theta = 1.$$

Moreover, it is easily seen that

$$\sum_{1\le|n|\le N} \frac{\overline{\chi}(n)}{n} e(n\alpha) \left(1 - \frac{|n|}{N}\right) = \sum_{1\le|n|\le N} \frac{\overline{\chi}(n)}{n} e(n\alpha) + O(1).$$

We have therefore proved:

[1]To this end, it is useful to note the following identities:

$$\Phi_N(\theta) = \frac{1}{N}\frac{\sin^2 \pi N\theta}{\sin^2 \pi\theta} = \frac{1}{N}\sum_{n\le N} D_{n-1}(\theta)$$

where D_j is the the Dirichlet kernel discussed above.

Lemma 1. *Let* χ (mod q) *be any Dirichlet character. Then*

$$\max_{\substack{\theta\in[0,1]\\ N\le q}}\left|\sum_{1\le|n|\le N}\frac{\overline{\chi}(n)}{n}e(n\theta)\right| = \max_{\theta\in[0,1]}\left|\sum_{1\le|n|\le q}\frac{\overline{\chi}(n)}{n}e(n\theta)\right| + O(1)$$

2. Paley's Construction

Below, the parity of characters will play an important role; we hope to avoid future confusion with a word of caution right away. Recall that a character χ is said to be *even* if $\chi(-1)=1$, and *odd* if $\chi(-1)=-1$. This is somewhat unfortunate nomenclature, since any odd character has even order (and equivalently, any character of odd order is even). Note that the converse does not hold.

Given a primitive character χ (mod q), recall Pólya's fourier expansion:

$$S_\chi(t) := \sum_{n\le t}\chi(n) = \frac{\tau(\chi)}{2\pi i}\sum_{1\le|n|\le q}\frac{\overline{\chi}(n)}{n}\left(1-e\left(-\frac{nt}{q}\right)\right)+O(\log q)$$

Here as usual $\tau(\chi)$ denotes the Gauss sum, and $e(x):=e^{2\pi i x}$. It follows that for any primitive even character,

$$S_\chi(t) = \frac{\tau(\chi)}{\pi}\sum_{n\le q}\frac{\overline{\chi}(n)}{n}\sin\frac{2\pi nt}{q}+O(\log q).$$

In particular, we see that for any primitive even character

$$M(\chi) := \max_{t\le q}|S_\chi(t)| \asymp \sqrt{q}\max_{\theta\in[0,1)}\left|\sum_{n\le q}\frac{\chi(n)}{n}\sin 2\pi n\theta\right| + O(\log q).$$

It follows from Lemma 1 that for any even character χ (mod q),

$$\max_{\substack{\theta\in[0,1)\\ N\le q}}\left|\sum_{n\le N}\frac{\chi(n)}{n}\sin 2\pi n\theta\right| \le \max_{\theta\in[0,1]}\left|\sum_{n\le q}\frac{\chi(n)}{n}\sin 2\pi n\theta\right| + O(1).$$

Using quadratic reciprocity and the Chinese Remainder Theorem, Paley constructs an infinite family X of primitive, even, real characters, such that for each χ (mod q) $\in X$ there exists $N_\chi\le q$ with

(1) $\chi(p)=\chi_{-4}(p)$ for all primes $p\le N_\chi$, and
(2) $q\le 1+4\prod_{p\le N_\chi}p$.

It follows that for any $\chi \pmod q \in X$,

$$\begin{aligned} M(\chi) &\gg \sqrt{q} \max_{\substack{\theta\in[0,1) \\ N\le q}} \left| \sum_{n\le N} \frac{\chi(n)}{n} \sin 2\pi n\theta \right| + O(\sqrt{q}) \\ &\ge \sqrt{q} \left| \sum_{n\le N_\chi} \frac{\chi(n)}{n} \sin \frac{\pi n}{2} \right| + O(\sqrt{q}) \\ &= \sqrt{q} \sum_{n\le N_\chi} \frac{\chi_{-4}(n)^2}{n} + O(\sqrt{q}) \\ &\gg \sqrt{q} \log N_\chi \\ &\gg \sqrt{q} \log\log q. \end{aligned}$$

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