Hölder Inequality as a Convexity Result

(and other ETS proofs)

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Henceforth, ETS = Enough To Show.

We warm up by proving the triangle inequality:

Proposition 1 (Triangle Inequality). *Given* $\{c_n\} \subset \mathbb{C}$*,*

$$\left|\sum c_n\right| \leq \sum |c_n|$$

Proof.
The case when $\sum c_n = 0$ is clearly true, so ETS that if $\sum c_n \neq 0$, then

$$\sum \left|\frac{c_n}{\sum c_k}\right| \geq 1$$

Letting $\alpha_n = \frac{c_n}{\sum c_k}$, it becomes ETS that if $\sum \alpha_n = 1$ then $\sum |\alpha_n| \geq 1$. But at this point we can stop ETSing and just prove this. Write $\alpha_n = r_n e^{i\theta_n}$. $\sum \alpha_n = 1$ means that $\sum r_n \cos\theta_n = 1$, whence

$$\begin{aligned} 1 &= \sum r_n \cos\theta_n \\ &\leq \sum r_n \\ &= \sum |\alpha_n| \end{aligned}$$

□

In a similar vein, we now prove Hölder's Inequality. This was explained to the author by Hugh Montgomery in MATH 775, during Spring 2006 at the University of Michigan.

Theorem 2 (Hölder's Inequality). *Given* $a_n, b_n \in \mathbb{C}$*, and* p, q *positive reals such that* $1/p + 1/q = 1$*. Then*

$$\left|\sum_n a_n b_n\right| \leq \left(\sum_k |a_k|^p\right)^{\frac{1}{p}} \left(\sum_k |b_k|^q\right)^{\frac{1}{q}}$$

Proof.
By the triangle inequality, ETS that

$$\sum_n |a_n b_n| \leq \left(\sum_k |a_k|^p\right)^{\frac{1}{p}} \left(\sum_k |b_k|^q\right)^{\frac{1}{q}}$$

Let $A_n = |a_n|^p$ and $B_n = |b_n|^q$. So ETS

$$\sum_n A_n^{1/p} B_n^{1/q} \leq \left(\sum_k A_k\right)^{\frac{1}{p}} \left(\sum_k B_k\right)^{\frac{1}{q}}$$

whence ETS

$$\sum_n \left(\frac{A_n}{\sum A_k}\right)^{\frac{1}{p}} \left(\frac{B_n}{\sum B_k}\right)^{\frac{1}{q}} \leq 1$$

Let $\sigma = 1/p$. Then $1/q = 1 - \sigma$, so ETS that for all $\sigma \in [0, 1]$,

$$\sum_n \left(\frac{A_n}{\sum A_k}\right)^{\sigma} \left(\frac{B_n}{\sum B_k}\right)^{1-\sigma} \leq 1$$

Let

$$\alpha_n = \frac{A_n}{\sum A_k} \qquad \beta_n = \frac{B_n}{\sum B_k}$$

Then ETS that $\forall \sigma \in [0, 1]$ and $\forall \alpha_n, \beta_n$ positive such that $\sum \alpha_n = \sum \beta_n = 1$,

$$f(\sigma) := \sum_n \beta_n \left(\frac{\alpha_n}{\beta_n}\right)^{\sigma} \leq 1$$

Since $f(0) = f(1) = 1$, ETS that $f(\sigma)$ is concave up on $[0, 1]$. Since $\beta_n > 0$, ETS

$$f_n(\sigma) := \left(\frac{\alpha_n}{\beta_n}\right)^{\sigma}$$

is concave up on $[0, 1]$, whence ETS that $g(x) := a^x$ is concave up on $[0, 1]$, where a is any positive constant. Finally, we're done ETSing: $g''(x) = a^x (\log a)^2 \geq 0$.

$\square$