NOTES ON ERROR ANALYSIS FROM LOOK-UP TABLES

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ABSTRACT. We show that the average squared error made in using a look-up table to evaluate cosine (where we always approximate by rounding down) is $2\pi - N\sin(2\pi/N) \approx \frac{4\pi^3}{3N^2}$, where N is the number of index points (i.e., the number of entries in the look-up table). This leads to an average error (per angle) of approximately $2\pi^2/3N^2$. Similar techniques should yield corresponding results for using the closest look-up table angle, as well as using a linear interpolation.

1. INTRODUCTION

The goal of this note is to compute, in closed form, the error in using a look-up table in evaluating cosine. We'll look at the mean square deviation, dividing the interval $[0, 2\pi)$ into N equal pieces. To keep the computation simple for now, we truncate the angle θ to the nearest angle in the look-up table.

Let us denote the N values in the look-up table by $\theta_n = 2\pi n/N$, where $n \in \{0, 1, \dots, N-1\}$ (and of course $\theta_N = 2\pi$). Thus we must compute

$$E(N) \ := \ \sum_{n=0}^{N-1} \int_{\theta_n}^{\theta_{n+1}} \left(\cos\theta - \cos\theta_n\right)^2 d\theta. \tag{1.1}$$

For now, we assume that any value we look up is known with complete accuracy. Our main result is

Theorem 1.1. *Consider a look-up table for cosine with N entries. Assuming each value of θ is equally likely, and assuming we use simple truncation, the sum of the integral of the absolute value of the difference squared has a nice expression:*

$$E(N) \ = \ 2\pi - N\sin(2\pi/N) \ \approx \ \frac{4\pi^3}{3N^2}. \tag{1.2}$$

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2. Evaluating error $E(N)$

We now determine the error $E(N)$. We first split the integral into three pieces:

$$\begin{aligned}
E(N) &= \sum_{n=0}^{N-1} \int_{\theta_n}^{\theta_{n+1}} (\cos\theta - \cos\theta_n)^2\, d\theta \\
&= \sum_{n=0}^{N-1} \int_{\theta_n}^{\theta_{n+1}} \left[\cos^2\theta - 2\cos\theta_n \cos\theta + \cos^2\theta_n\right] d\theta \\
&= \sum_{n=0}^{N-1} \int_{\theta_n}^{\theta_{n+1}} \cos^2\theta d\theta - 2\sum_{n=0}^{N-1} \cos\theta_n \int_{\theta_n}^{\theta_{n+1}} \cos\theta d\theta + \sum_{n=0}^{N-1} \cos^2\theta_n \int_{\theta_n}^{\theta_{n+1}} d\theta \\
&= \int_0^{2\pi} \cos^2\theta d\theta - 2\sum_{n=0}^{N-1} \cos\theta_n \left(\sin\theta_{n+1} - \sin\theta_n\right) + \frac{2\pi}{N}\sum_{n=0}^{N-1} \cos^2\theta_n \\
&= \pi + \frac{2\pi}{N}\sum_{n=0}^{N-1} \cos^2\theta_n - 2\sum_{n=0}^{N-1} \cos\theta_n \left(\sin\theta_{n+1} - \sin\theta_n\right) \\
&= \pi + T_2 + T_3, \qquad (2.1)
\end{aligned}$$

where we used $\int_0^{2\pi} \sin^2\theta d\theta = \int_0^{2\pi} \cos^2\theta d\theta = \frac{1}{2}\int_0^{2\pi} d\theta$.

We now handle the other two terms. The sum of the cosine-squared term is readily handled. We may extend the sum to be from $-(N-1)$ to $N-1$ at the cost of halving the result after adding in the $n = 0$ term, which is only counted once. We then use $\cos\theta_n = (\exp(j\theta_n) + \exp(-j\theta_n))/2$, where $j = \sqrt{-1}$. This yields

$$\begin{aligned}
T_2 &= \frac{2\pi}{N}\sum_{n=0}^{N-1} \cos^2\theta_n \\
&= \frac{\pi}{N} + \frac{\pi}{N}\sum_{n=-(N-1)}^{N-1} \cos^2\theta_n \\
&= \frac{\pi}{N} + \frac{\pi}{N}\sum_{n=-(N-1)}^{N-1} \frac{1}{4}\left(e^{2j\theta_n} + 2 + e^{-2j\theta_n}\right) \\
&= \frac{\pi}{N} + \frac{\pi}{2N}(2N-1) + \frac{\pi}{4N}\sum_{n=-(N-1)}^{N-1} \left(e^{4\pi jn/N} + e^{-4\pi jn/N}\right) \\
&= \pi + \frac{\pi}{2N} - \frac{\pi}{4N}\left[\sum_{n=-(N-1)}^{N-1} (\exp(4\pi j/N))^n + \sum_{n=-(N-1)}^{N-1} (\exp(-4\pi j/N))^n\right]. \qquad (2.2)
\end{aligned}$$

We now have two geometric series. Note that they would vanish if we were summing from $-N$ to $N-1$. Thus the value of the two sums is simply their ratio evaluated at $-N$ (in other words, extend the sum to start at $-N$ and then subtract this added term); as each is the complex conjugate of the other, we get simply twice the real part, so

$$T_2 \;=\; \pi + \frac{\pi}{2N} - \frac{\pi}{4N}\left[-2\cos(4\pi(-N)/N)\right] \;=\; \pi. \qquad (2.3)$$

Numerical computations confirm this value.

We thus have

$$E(N) \;=\; \pi + \pi + T_3 \;=\; 2\pi + T_3. \tag{2.4}$$

We now work on simplifying T_3. We have

$$\sin\theta_{n+1} \;=\; \sin\left(\theta_n + \frac{2\pi}{N}\right) \;=\; \sin\theta_n \cos\frac{2\pi}{N} + \cos\theta_n \sin\frac{2\pi}{N}; \tag{2.5}$$

the advantage of this is that the cosine and sine of $2\pi/N$ is constant (for fixed N), and note that $2\pi/N = \theta_1$. This implies

$$\sin\theta_{n+1} - \sin\theta_n \;=\; (\cos\theta_1 - 1)\sin\theta_n + \sin\theta_1\cos\theta_n. \tag{2.6}$$

Thus

$$\begin{aligned}
T_3 &= -2\sum_{n=0}^{N-1}\cos\theta_n\left(\sin\theta_{n+1} - \sin\theta_n\right)\\
&= -2\sum_{n=0}^{N-1}\cos\theta_n\left[(\cos\theta_1 - 1)\sin\theta_n + \sin\theta_1\cos\theta_n\right]\\
&= -2\left(\cos\theta_1 - 1\right)\sum_{n=0}^{N-1}\cos\theta_n\sin\theta_n - 2\sin\theta_1\sum_{n=0}^{N-1}\cos^2\theta_n\\
&= -\left(\cos\theta_1 - 1\right)\sum_{n=0}^{N-1}\sin(2\theta_n) - \frac{2N\sin\theta_1}{2\pi}\frac{2\pi}{N}\sum_{n=0}^{N-1}\cos^2\theta_n\\
&= -\left(\cos\theta_1 - 1\right)\sum_{n=0}^{N-1}\sin(2\theta_n) - \frac{2N\sin\theta_1}{2\pi}\cdot\pi, \qquad (2.7)
\end{aligned}$$

where the last follows from the fact that our sum is just T_2, which we showed equals π. What about the sum of the sine term? We can evaluate that easily by using $\sin\theta = (\exp(j\theta) - \exp(-j\theta))/2i$. Substituting this in and using $\theta_n = 2\pi n/N$ yields

$$\begin{aligned}
T_3 &= -\left(\cos\theta_1 - 1\right)\sum_{n=0}^{N-1}\frac{\exp(4\pi jn/N) - \exp(-4\pi jn/N)}{2i} - N\sin\theta_1\\
&= -\frac{\cos\theta_1 - 1}{2i}\sum_{n=0}^{N-1}\left[\exp(4\pi j/N)^n - \exp(-4\pi j/N)^n\right] - N\sin\theta_1\\
&= -N\sin\theta_1, \qquad (2.8)
\end{aligned}$$

as the two geometric series vanish.

Thus, putting all the pieces together, we have shown

$$E(N) \;=\; 2\pi - N\sin\theta_1. \tag{2.9}$$

For small angles, $\sin\theta \approx \theta - \theta^3/3!$. Noting $\theta_1 = 2\pi/N$, we see

$$E(N) \;\approx\; 2\pi - N\left[\frac{2\pi}{N} - \frac{8\pi^3}{3!N^3} - \cdots\right] \;\approx\; \frac{4\pi^3}{3N^2}. \tag{2.10}$$

This completes the proof of our main result. □

Though it is not needed, we did perform some numerical evaluations of $E(N)$ for various N, comparing the unsimplified sum to our expression. As was to be expected, our investigations showed excellent agreement with the numerics.

Another item: it might be worthwhile to consider, not the quantity $E(N)$ but instead $E(N)/2\pi$. The reason is that $E(N)$ is the total error, while $E(N)/2\pi$ is the average error per angle. This would give an average error of approximately $2\pi^2/3N^2$.

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