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Department of Physics

Level 3 Mathematical Methods 33010 (2009/2010)

Definite Integrals by Contour Integration

Many kinds of (real) definite integrals can be found using the results we have found for contour integrals in the complex plane. This is because the values of contour integrals can usually be written down with very little difficulty. We simply have to locate the poles inside the contour, find the residues at these poles, and then apply the residue theorem. The more subtle part of the job is to choose a suitable contour integral i.e. one whose evaluation involves the definite integral required. We illustrate these steps for a set of five types of definite integral.

Type 1 Integrals

Integrals of trigonometric functions from 0 to 2π:

$$I = \int_0^{2\pi} (\text{trig function}) d\theta$$

By "trig function" we mean a function of $\cos\theta$ and $\sin\theta$.
The obvious way to turn this into a contour integral is to choose the unit circle as the contour, in other words to write $z = \exp i\theta$, and integrate with respect to θ. *On the unit circle* , both $\cos\theta$ and $\sin\theta$ can be written as simple algebraic functions of z:

$$\cos\theta = \frac{1}{2}(z + 1/z) \qquad \sin\theta = \frac{1}{2i}(z - 1/z)$$

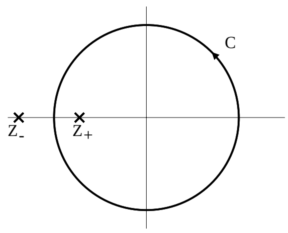
and making this replacement turns the trigonometric function into an algebraic function of z whose poles can be easily found.
Example:

$$I = \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} \qquad \text{where } -1 < a < +1$$

$$I = \oint \frac{1}{1 + \frac{a}{2}(z + \frac{1}{z})} \frac{dz}{iz} = \frac{2}{i} \oint \frac{dz}{2z + az^2 + a}$$

The poles of the function being integrated lie at the roots of the equation $az^2 + 2z + a = 0$ i.e. at the points

$$z_\pm = \frac{1}{a}\left(-1 \pm \sqrt{1 - a^2}\right)$$



Of the poles, only z_+ lies inside the unit circle, so $I = 2\pi i R_+$ where R_+ is the residue at z_+ To find the residue we note that this is a simple pole and if we write the integrand as $f(z) = g(z)/h(z)$ the residue at z_+ is:

$$\frac{g(z_+)}{h'(z_+)} = \frac{2}{2i(az_+ + 1)} = \frac{1}{i\sqrt{1-a^2}}$$

Hence the integral required is $2\pi/\sqrt{1-a^2}$

Type 2 Integrals

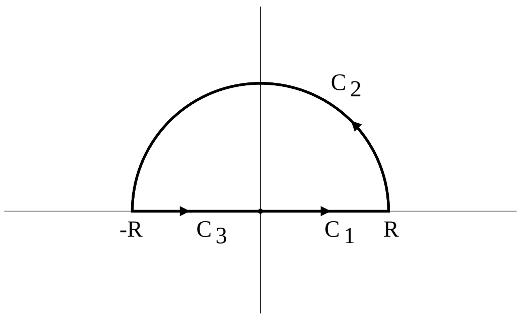
Integrals such as

$$I = \int_{-\infty}^{+\infty} f(x)dx$$

or, equivalently, in the case where $f(x)$ is an even function of x

$$I = \int_{0}^{+\infty} f(x)dx$$

can be found quite easily, by inventing a closed contour in the complex plane which includes the required integral. The simplest choice is to close the contour by a very large semi-circle in the upper half-plane. Suppose we use the symbol "R" for the radius. The entire contour integral comprises the integral along the real axis from $-R$ to $+R$ together with the integral along the semi-circular arc. In the limit as $R \to \infty$ the contribution from the straight line part approaches the required integral I, while the curved section may in some cases vanish in the limit. Note: when we



choose to split up the complete contour into sections (C_1, C_2, etc.) we shall use the notation (J_1, J_2, etc.) for the corresponding contour integrals. i.e.

$$J_1 = \int_{C_1} f(z)dz \qquad\qquad \text{etc.}$$

Example:

$$I = \int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^2}$$

can be found by these methods, because the integral around the arc, which can be written as

$$\int_0^{\pi} \frac{iRe^{i\theta}d\theta}{(R^2e^{2i\theta}+1)^2} \xrightarrow{R\to\infty} \int_0^{\pi} \frac{iRe^{i\theta}d\theta}{R^4e^{4i\theta}} = \frac{i}{R^3}\int_0^{\pi} e^{-3i\theta}d\theta$$

clearly vanishes as $R \to \infty$. Once again the way is clear for us to use the residue theorem, and inspection of the function

$$\frac{1}{(z^2+1)^2}$$

shows that it has poles at the roots of $z^2+1=0$ i.e. $z=\pm i$, of which only $z=+i$ lies in the upper half-plane. The order of the pole is established by noting that $(z-i)f(z)|_{z=i}$ is infinite, while $(z-i)^2 f(z)|_{z=i} = \frac{1}{(z+i)^2} = -\frac{1}{4}$, so the pole is of second order.
Finally to find the residue for the pole we have to use the general formula

$$R = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)$$

where $m \geq N$. Since $N=2$ in this case the simplest choice is $m=N=2$, giving

$$R = \lim_{z\to i} \frac{d}{dz}\left\{\frac{1}{(z+i)^2}\right\} = -\frac{i}{4}$$

Consequently $I = 2\pi i \times \frac{-i}{4} = \pi/2$

Type 3 Integrals

This class is similar to the previous one, but with a trigonometric function involved in the integrand:

$$I = \int_{-\infty}^{+\infty} \frac{\text{trig fn}}{\text{polynomial}} dx$$

In this case we have to take special care over the choice of the complex function, in other words the *continuation* of the trigonometric function away from the real axis. The three functions $\cos z$, (e^{iz}) and (e^{-iz}) all have the same real parts on the real axis, but are different elsewhere. In particular, when $z=iR$ and $R\to\infty$ the first and last become infinite, while the second tends to zero. Consequently the methods described for Type 2 integrals will work only if we adopt the second continuation.

Example:

$$I = \int_{-\infty}^{+\infty} \frac{\cos x dx}{x^2+a^2}$$

For the reasons just described we find this by contour integration of

$$\frac{e^{iz}}{z^2+a^2}$$

and since in polar coordinates $e^{iz} = e^{ir\cos\theta}e^{-r\sin\theta}$ the numerator tends to zero as r becomes large everywhere in the upper half-plane where $\sin\theta$ is positive. Using the same D-shaped contour as before, the semi-circular arc contributes

$$\int_{arc} \frac{e^{iz}dz}{z^2+a^2} = \lim_{R\to\infty}\int_0^{+\pi} \frac{e^{iz}iRe^{i\theta}d\theta}{R^2e^{2i\theta}+a^2} = 0$$

So $I = 2\pi i\Sigma R$ where the sum is of the residues in the upper half-plane. The function has simple poles at $z = \pm ia$ of which $z = +ia$ is in the upper half-plane with residue $R = (z-ia)f(z)|_{z=ia} = \frac{e^{iz}}{z+ia}|_{z=ia} = -\frac{i}{2a}e^{-a}$ So finally, taking real parts, $I = \frac{\pi}{a}e^{-a}$. This argument will clearly work whenever the integral around the semi-circle vanishes. The previous discussion shows this is true for functions of the form

$$f(z) = \frac{P(z)}{Q(z)}e^{iz}$$

where P, Q are polynomials in z and the order of Q is at least 2 greater than that of P, because in the limit of large R, the contribution from the arc will fall as $1/R$ (or a higher inverse power). If, on the other hand, the order of Q is just one higher than that of P, it is not immediately clear what will happen. However, in such cases the exponential decrease of $e^{-R\sin\theta}$ in the upper half-plane overwhelms everything else and the arc integral still vanishes. This fact, which we do not have time to prove formally, is known as "Jordan's Lemma" and it makes contour integration a useful method for a large class of integrals, and you should know it and be ready to use it in appropriate cases.

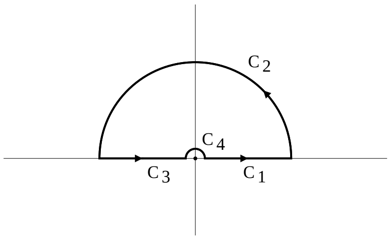
Type 4 Integrals

A type of integral which brings in some new ideas is similar to Type 2 but with a pole of the integrand actually on the contour of integration. As an example of a situation where this arises, consider the real integral

$$I = \int_{-\infty}^{+\infty} \frac{\cos x}{x}dx$$

The approach previously discussed would involve replacing $\cos x/x$ by e^{iz}/z , in which case the semi-circular arc would vanish by Jordan's lemma. However there is a problem, because $\cos x/x$ has a pole at the origin, and the integrand diverges as we approach the origin along the real axis either from positive or negative values of z. To deal with this we introduce the concept of the "Principal Value" of a definite integral. In this we imagine that we exclude a small *symmetrical* range of real z values around the pole, and take the limit as this excluded region shrinks to zero width. We can find this by a suitable contour integral.

To do this in our example we find the contour integral of e^{iz}/z around a contour similar to that used above, but also involving a small semi-circular detour around the pole at the origin: There are no poles inside this contour so the total contour integral vanishes ($J_1 + J_2 + J_3 + J_4 = 0$). The integral around the big semi-circle (J_2) also vanishes in the limit of large radius, and the integral along the real axis



$(J_3 + J_1)$ is what we have just defined as a Principal Value in the limit as $r \to 0$ and $R \to \infty$. To find the contribution from the small semi-circle (J_4) we evaluate

$$J_4 = \lim_{r\to 0} \int_{-\pi}^{0} \frac{e^{ire^{i\theta}} ire^{i\theta} d\theta}{re^{i\theta}} = i \int_{-\pi}^{0} 1 d\theta = -\pi i$$

The vanishing of the entire contour integral yields

$$0 = J_1 + J_2 + J_3 + J_4 = PV \int_{-\infty}^{+\infty} \frac{e^{ix} dx}{x} + 0 - \pi i$$

So we can separate real and imaginary parts to obtain

$$PV \int_{-\infty}^{+\infty} \frac{\cos x}{x} dx = 0 \qquad\qquad \int_{-\infty}^{+\infty} \frac{\sin x}{x} dx = \pi$$

(We have omitted the "PV" in the final integral because $\frac{\sin x}{x}$ is actually finite at $x = 0$.)

Type 5 Integrals

Our last type of integral will be those involving *branch cuts*. Far from being a problem, these can actually make some kinds of definite integral possible because we can make use of the discontinuity across the cut to construct the required integral. This is best shown by an example:

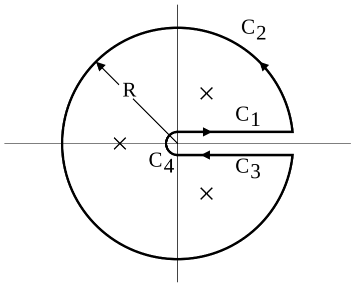
Example

$$I = \int_{0}^{+\infty} \frac{dx}{x^3 + 1}$$

resembles Type 2, but because the integrand is not even we cannot extend the integration to the whole real axis and then halve the result. However, suppose we look at the contour integral

$$J = \int_C \frac{\ln z dz}{z^3 + 1}$$

around the contour shown. Note that this contour does not pass through the cut onto another branch of the function. Remember that $\ln z = \ln r + i\theta + 2\pi i n$ where n is an integer distinguishing the branches of the function. On our contour we have points just above the cut in the section C_1 and points at the same x values but just below the cut in C_3. Because we stay on the same sheet (say $n = 0$) throughout our contour, these values differ by $2\pi i$. The sections C_1 and C_3 are described in opposite



directions, so it is just the difference in the values of the integrand that contributes to J. In the limit of large radius the contribution of C_2 vanishes, as also does C_4 in the limit of *small* radius. There is therefore a simple connection between J and the original definite integral I:

$$J = 2\pi i I$$

On the other hand we can find the value of J from the residue theorem, since $\ln z/(z^3+1)$ has three simple poles at the cube roots of -1 , which are $e^{i\pi/3}$,$e^{i\pi}$ and $e^{5i\pi/3}$. To find the residues we can use the $g(z_0)/h'(z_0)$ method, with $g(z) = \ln z$ and $h(z) = z^3 + 1$. This gives the values

$$\frac{i\pi}{9}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right), \frac{i\pi}{3}, \frac{5i\pi}{9}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$$

The sum of these is $-2\pi/3\sqrt{3}$. So by the residue theorem, $J = 2\pi i \times \left(-\frac{2\pi}{3\sqrt{3}}\right)$ and $I = 2\pi/3\sqrt{3}$

updated R. Tapper November 26, 2006