Intermediate and Mean Value Theorems and Taylor Series

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Abstract

Using just the Mean Value Theorem, we prove the n^{th} Taylor Series Approximation. Namely, if f is differentiable at least $n+1$ times on $[a,b]$, then $\forall x \in [a,b]$, $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ plus an error that is at most $\max_{a \le c \le x} |f^{(n+1)}(c)| \cdot |x-a|^{n+1}$.

1 Mean Value Theorem

Let $h(x)$ be differentiable on $[a,b]$, with continuous derivative. Then

$$h(b) - h(a) = h'(c) \cdot (b-a), \qquad c \in [a,b]. \tag{1}$$

The MVT follows immediately from the Intermediate Value Theorem: Let f be a continuous function on $[a,b]$. $\forall C$ between $f(a)$ and $f(b)$, $\exists c \in [a,b]$ such that $f(c) = C$. In other words, all intermediate values of a continuous function are obtained. We will sketch a proof later.

2 Notation

$[a,b] = \{x : a \le x \le b\}$. IE, $[a,b]$ is all x between a and b, including a and b. $(a,b) = \{x : a < x < b\}$. IE, (a,b) is all x between a and b, not including the endpoints a and b.

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3 Taylor Series

Assuming f is differentiable $n+1$ times on $[a,b]$, we apply the MVT multiple times to bound the error between $f(x)$ and its Taylor Approximations.

Let

$$\begin{aligned} f_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ h(x) &= f(x) - f_n(x). \end{aligned} \tag{2}$$

$f_n(x)$ is the n^{th} Taylor Series Approximation to $f(x)$. Note $f_n(x)$ is a polynomial of degree n.

We want to bound $|h(x)|$ for $x \in [a,b]$. Without loss of generality (basically, for notational convenience), we may assume $a = 0$ and $f(a) = 0$.

Thus, $h(0) = 0$. Applying the MVT to h yields

$$\begin{aligned} h(x) &= h(x) - h(0) \\ &= h'(c_1) \cdot (x - 0) \\ &= \Big(f'(c_1) - f_n'(c_1)\Big)x \\ &= \Big(f'(c_1) - \sum_{k=1}^{n} \frac{f^{(k)}(0)}{k!} \cdot k(c_1 - 0)^{k-1}\Big)x \\ &= \Big(f'(c_1) - \sum_{k=1}^{n} \frac{f^{(k)}(0)}{(k-1)!} c_1^{k-1}\Big)x \\ &= h_1(c_1)x. \end{aligned} \tag{3}$$

We now apply the MVT to $h_1(u)$. Note that $h_1(0) = 0$. Therefore

$$\begin{aligned} h_1(c_1) &= h_1(c_1) - h_1(0) \\ &= h_1'(c_2) \cdot (c_1 - 0) \\ &= \Big(f''(c_2) - f_n''(c_2)\Big)c_1 \\ &= \Big(f''(c_2) - \sum_{k=2}^{n} \frac{f^{(k)}(0)}{(k-1)!} \cdot (k-1)(c_2 - 0)^{k-2}\Big)c_1 \\ &= \Big(f''(c_2) - \sum_{k=2}^{n} \frac{f^{(k)}(0)}{(k-2)!} c_2^{k-2}\Big)c_1 \\ &= h_2(c_1)c_1. \end{aligned} \tag{4}$$

Therefore,

$$h(x) = f(x) - f_n(x) = h_2(c_2)c_1x, \quad c_2 \in [0, c_1],\ c_1 \in [0, x]. \tag{5}$$

Proceeding in this way a total of n times yields

$$h(x) \ = \ \Big(f^{(n)}(c_n) - f^{(n)}(0)\Big)c_{n-1}c_{n-2}\cdots c_2c_1x. \tag{6}$$

Applying the MVT to $f^{(n)}(c_n) - f^{(n)}(0)$ gives $f^{(n+1)}(c_{n+1}) \cdot (c_n - 0)$. Thus,

$$h(x) = f(x) - f_n(x) = f^{(n+1)}(c_{n+1})c_n \cdots c_1x, \quad c_i \in [0, x]. \tag{7}$$

Therefore

$$|h(x)| = |f(x) - f_n(x)| = M_{n+1}|x|^{n+1}, \quad M_{n+1} = \max_{c \in [0,x]} |f^{(n+1)}(c)|. \tag{8}$$

Thus, if f is differentiable $n+1$ times, the n^{th} Taylor Series Approximation to $f(x)$ is correct within a multiple of $|x|^{n+1}$; further, the multiple is bounded by the maximum value of $f^{(n+1)}$ on $[0, x]$.

4 Sketch of Proof of the MVT

The MVT follows from Rolle's Theorem: Let f be differentiable on $[a, b]$, and assume $f(a) = f(b) = 0$. Then there exists a $c \in [a, b]$ such that $f'(c) = 0$.

Why? Assume Rolle's Theorem. Consider the function

$$h(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a). \tag{9}$$

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Note $h(a) = f(a) - f(a) = 0$ and $h(b) = f(b) - (f(b) - f(a)) - f(a) = 0$. Thus, the conditions of Rolle's Theorem are satisfied for $h(x)$, and there is some $c \in [a, b]$ such that $h'(c) = 0$. But

$$h'(c) \ = \ f'(c) - \frac{f(b) - f(a)}{b - a}. \tag{10}$$

Rewriting yields $f(b) - f(a) = f'(c) \cdot (b - a)$.

Thus, it is sufficient to prove Rolle's Theorem to prove the MVT.

Without loss of generality, assume $f'(a)$ and $f'(b)$ are non-zero. If either were zero, we would be done.

Multiplying $f(x)$ by -1 if needed, we may assume $f'(a) > 0$.

Case 1: $f'(b) < 0$: As $f'(a) > 0$ and $f'(b) < 0$, the Intermediate Value Theorem, applied to $f'(x)$, asserts that all intermediate values are attained. As $f'(b) < 0 < f'(a)$, this implies the existence of a $c \in (a, b)$ such that $f'(c) = 0$.

Case 2: $f'(b) > 0$: $f(a) = f(b) = 0$, and the function f is increasing at a and b. If x is real close to a, then $f(x) > 0$ because $f'(a) > 0$.

This follows from the fact that

$$f'(0) \quad = \quad \lim_{x \to 0} \frac{f(x) - f(0)}{x}. \tag{11}$$

As $f'(0) > 0$, the limit is positive. As the denominator is positive for $x > 0$, the numerator must be positive. Thus, $f(x)$ must be greater than $f(0)$ for small x.

Similarly, $f'(b) > 0$ implies $f(x) < f(b) = 0$ for x near b.

Therefore, the function $f(x)$ is positive for x slightly greater than a and negative for x slightly less than b. If the first derivative were always positive, then $f(x)$ could never be negative as it starts at 0 at a. This can be seen by again using the limit definition of the first derivative to show that if $f'(x) > 0$, then the function is increasing near x. See the next section for more details.

Thus, the first derivative cannot always be positive. Either there must be some point $y \in (a, b)$ such that $f'(y) = 0$ (and we are then done!) or $f'(y) < 0$. By the IVT, as 0 is between $f'(a)$ (which is positive) and $f'(y)$ (which is negative), there is some $c \in (a, y) \subset [a, b]$ such that $f'(c) = 0$.

5 Sign of the Derivative

As it is such an important concept, let us show that $f'(x) > 0$ implies $f(x)$ is increasing at x. The definition of the derivative gives

$$f'(x) \quad = \quad \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{12}$$

If $\Delta x > 0$, the denominator is positive. As the limit is positive, for Δx sufficiently small, the numerator must be positive. Thus, Δx positive and small implies $f(x + \Delta x) > f(x)$.

If $\Delta x < 0$, the denominator is negative. As the limit is positive, for Δx sufficiently small, the numerator must be negative. Thus, Δx negative and small implies $f(x + \Delta x) < f(x)$.

Therefore, if $f'(x)$ is positive, then f is increasing at x. Similarly we can show if $f'(x)$ is negative then f is decreasing at x.

6 Intermediate Value Theorem

We have reduced all our proofs to the intuitively plausible IVT: if C is between $f(a)$ and $f(b)$ for some continuous function f, then $\exists c \in (a, b)$ such that $f(c) = C$.

Here is a sketch of a proof using the method Divide and Conquer. Without loss of generality, assume $f(a) < C < f(b)$. Let x_1 be the midpoint of $[a, b]$. If $f(x_1) = C$ we are done. If $f(x_1) < C$, we look at the interval $[x_1, b]$. If $f(x_1) > C$ we look at the interval $[a, x_1]$.

In either case, we have a new interval, call it $[a_1, b_1]$, such that $f(a_1) < C < f(b_1)$, and the interval has size half that of $[a, b]$. Continuing in this manner, constantly taking the midpoint and looking at the appropriate half-interval, we see one of two things may happen.

First, we may be lucky and one of the midpoints may satisfy $f(x_n) = C$. In this case, we have found the desired point c.

Second, no midpoint works. Thus, we divide infinitely often, getting a sequence of points x_n. This is where rigorous mathematical analysis is required.

We claim the sequence of points x_n converge to some number $X \in (a, b)$. Clearly it can't be an endpoint. We keep getting smaller and smaller intervals (of half the size of the previous and contained in the previous) where $f(x) < C$ at the left endpoint, and $f(x) > C$ at the right endpoint. By continuity at the point X, eventually $f(x)$ must be close to $f(X)$ for x close to X.

If $f(X) < C$, then eventually the right endpoint cannot be greater than C; if $f(X) > C$, eventually the left endpoint cannot be less than C. Thus, $f(X) = C$.