

# 1 Preliminaries to Complex Analysis

The sweeping development of mathematics during the last two centuries is due in large part to the introduction of complex numbers; paradoxically, this is based on the seemingly absurd notion that there are numbers whose squares are negative.

*E. Botti, 1952*

This chapter is devoted to the exposition of basic preliminary material which we use extensively throughout of this book.

We begin with a quick review of the algebraic and analytic properties of complex numbers followed by some topological notions of sets in the complex plane. (See also the exercises at the end of Chapter 1 in Book I.)

Then, we define precisely the key notion of holomorphicity, which is the complex analytic version of differentiability. This allows us to discuss the Cauchy-Riemann equations, and power series.

Finally, we define the notion of a curve and the integral of a function along it. In particular, we shall prove an important result, which we state loosely as follows: if a function  $f$  has a primitive, in the sense that there exists a function  $F$  that is holomorphic and whose derivative is precisely  $f$ , then for any closed curve  $\gamma$

$$\int_{\gamma} f(z) dz = 0.$$

This is the first step towards Cauchy's theorem, which plays a central role in complex function theory.

## 1 Complex numbers and the complex plane

Many of the facts covered in this section were already used in Book I.

### 1.1 Basic properties

A complex number takes the form  $z = x + iy$  where  $x$  and  $y$  are real, and  $i$  is an imaginary number that satisfies  $i^2 = -1$ . We call  $x$  and  $y$  the

**real part** and the **imaginary part** of  $z$ , respectively, and we write

$$x = \operatorname{Re}(z) \quad \text{and} \quad y = \operatorname{Im}(z).$$

The real numbers are precisely those complex numbers with zero imaginary parts. A complex number with zero real part is said to be **purely imaginary**.

Throughout our presentation, the set of all complex numbers is denoted by  $\mathbb{C}$ . The complex numbers can be visualized as the usual Euclidean plane by the following simple identification: the complex number  $z = x + iy \in \mathbb{C}$  is identified with the point  $(x, y) \in \mathbb{R}^2$ . For example, 0 corresponds to the origin and  $i$  corresponds to  $(0, 1)$ . Naturally, the  $x$  and  $y$  axes of  $\mathbb{R}^2$  are called the **real axis** and **imaginary axis**, because they correspond to the real and purely imaginary numbers, respectively. (See Figure 1.)

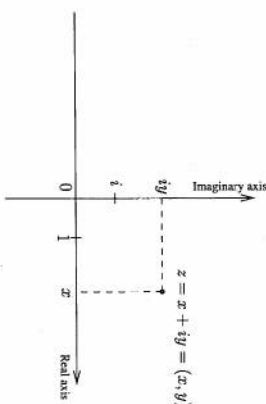


Figure 1. The complex plane

The natural rules for adding and multiplying complex numbers can be obtained simply by treating all numbers as if they were real, and keeping in mind that  $i^2 = -1$ . If  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$ , then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2),$$

and also

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1 x_2 + ix_1 y_2 + iy_1 x_2 + i^2 y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2). \end{aligned}$$

If we take the two expressions above as the definitions of addition and multiplication, it is a simple matter to verify the following desirable properties:

- Commutativity:  $z_1 + z_2 = z_2 + z_1$  and  $z_1 z_2 = z_2 z_1$  for all  $z_1, z_2 \in \mathbb{C}$ .
- Associativity:  $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ ; and  $(z_1 z_2) z_3 = z_1 (z_2 z_3)$  for  $z_1, z_2, z_3 \in \mathbb{C}$ .
- Distributivity:  $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$  for all  $z_1, z_2, z_3 \in \mathbb{C}$ .

Of course, addition of complex numbers corresponds to addition of the corresponding vectors in the plane  $\mathbb{R}^2$ . Multiplication, however, consists of a rotation composed with a dilation, a fact that will become transparent once we have introduced the polar form of a complex number. At present we observe that multiplication by  $i$  corresponds to a rotation by an angle of  $\pi/2$ .

The notion of length, or absolute value of a complex number is identical to the notion of Euclidean length in  $\mathbb{R}^2$ . More precisely, we define the **absolute value** of a complex number  $z = x + iy$  by

$$|z| = (x^2 + y^2)^{1/2},$$

so that  $|z|$  is precisely the distance from the origin to the point  $(x, y)$ . In particular, the triangle inequality holds:

$$|z + w| \leq |z| + |w| \quad \text{for all } z, w \in \mathbb{C}.$$

We record here other useful inequalities. For all  $z \in \mathbb{C}$  we have both  $|\operatorname{Re}(z)| \leq |z|$  and  $|\operatorname{Im}(z)| \leq |z|$ , and for all  $z, w \in \mathbb{C}$

$$||z| - |w|| \leq |z - w|.$$

This follows from the triangle inequality since

$$|z| \leq |z - w| + |w| \quad \text{and} \quad |w| \leq |z - w| + |z|.$$

The **complex conjugate** of  $z = x + iy$  is defined by

$$\bar{z} = x - iy,$$

and it is obtained by a reflection across the real axis in the plane. In fact a complex number  $z$  is real if and only if  $z = \bar{z}$ , and it is purely imaginary if and only if  $z = -\bar{z}$ .



The reader should have no difficulty checking that

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2} \quad \text{and} \quad \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}.$$

Also, one has

$$|z|^2 = z\bar{z} \quad \text{and as a consequence} \quad \frac{1}{z} = \frac{\bar{z}}{|z|^2} \quad \text{whenever } z \neq 0.$$

Any non-zero complex number  $z$  can be written in **polar form**

$$z = re^{i\theta},$$

where  $r > 0$ ; also  $\theta \in \mathbb{R}$  is called the **argument** of  $z$  (defined uniquely up to a multiple of  $2\pi$ ) and is often denoted by  $\arg z$ , and

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

Since  $|e^{i\theta}| = 1$  we observe that  $r = |z|$ , and  $\theta$  is simply the angle (with positive counterclockwise orientation) between the positive real axis and the half-line starting at the origin and passing through  $z$ . (See Figure 2.)

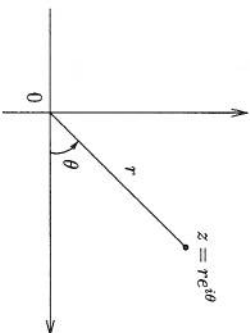


Figure 2. The polar form of a complex number

Finally, note that if  $z = re^{i\theta}$  and  $w = se^{i\varphi}$ , then

$$zw = rse^{i(\theta+\varphi)},$$

so multiplication by a complex number corresponds to a homothety in  $\mathbb{R}^2$  (that is, a rotation composed with a dilation).

## 1.2 Convergence

We make a transition from the arithmetic and geometric properties of complex numbers described above to the key notions of convergence and limits.

A sequence  $\{z_1, z_2, \dots\}$  of complex numbers is said to **converge** to  $w \in \mathbb{C}$  if

$$\lim_{n \rightarrow \infty} |z_n - w| = 0, \quad \text{and we write} \quad w = \lim_{n \rightarrow \infty} z_n.$$

This notion of convergence is not new. Indeed, since absolute values in  $\mathbb{C}$  and Euclidean distances in  $\mathbb{R}^2$  coincide, we see that  $z_n$  converges to  $w$  if and only if the corresponding sequence of points in the complex plane converges to the point that corresponds to  $w$ .

As an exercise, the reader can check that the sequence  $\{z_n\}$  converges to  $w$  if and only if the sequence of real and imaginary parts of  $z_n$  converge to the real and imaginary parts of  $w$ , respectively.

Since it is sometimes not possible to readily identify the limit of a sequence (for example,  $\lim_{n \rightarrow \infty} \sum_{k=1}^n 1/n^2$ ), it is convenient to have a condition on the sequence itself which is equivalent to its convergence. A sequence  $\{z_n\}$  is said to be a **Cauchy sequence** (or simply **Cauchy**) if

$$|z_n - z_m| \rightarrow 0 \quad \text{as } n, m \rightarrow \infty.$$

In other words, given  $\epsilon > 0$  there exists an integer  $N > 0$  so that  $|z_n - z_m| < \epsilon$  whenever  $n, m > N$ . An important fact of real analysis is that  $\mathbb{R}$  is complete: every Cauchy sequence of real numbers converges to a real number.<sup>1</sup> Since the sequence  $\{z_n\}$  is Cauchy if and only if the sequences of real and imaginary parts of  $z_n$  are, we conclude that every Cauchy sequence in  $\mathbb{C}$  has a limit in  $\mathbb{C}$ . We have thus the following result.

**Theorem 1.1**  *$\mathbb{C}$ , the complex numbers, is complete.*

We now turn our attention to some simple topological considerations that are necessary in our study of functions. Here again, the reader will note that no new notions are introduced, but rather previous notions are now presented in terms of a new vocabulary.

### 1.3 Sets in the complex plane

If  $z_0 \in \mathbb{C}$  and  $r > 0$ , we define the **open disc**  $D_r(z_0)$  of radius  $r$  centered at  $z_0$  to be the set of all complex numbers that are at absolute

<sup>1</sup>This is sometimes called the Bolzano-Weierstrass theorem.

value strictly less than  $r$  from  $z_0$ . In other words,

$$D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\},$$

and this is precisely the usual disc in the plane of radius  $r$  centered at  $z_0$ . The **closed disc**  $\bar{D}_r(z_0)$  of radius  $r$  centered at  $z_0$  is defined by

$$\bar{D}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\},$$

and the boundary of either the open or closed disc is the circle

$$C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}.$$

Since the **unit disc** (that is, the open disc centered at the origin and of radius 1) plays an important role in later chapters, we will often denote it by  $\mathbb{D}$ ,

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

Given a set  $\Omega \subset \mathbb{C}$ , a point  $z_0$  is an **interior point** of  $\Omega$  if there exists  $r > 0$  such that

$$D_r(z_0) \subset \Omega.$$

The **interior** of  $\Omega$  consists of all its interior points. Finally, a set  $\Omega$  is **open** if every point in that set is an interior point of  $\Omega$ . This definition coincides precisely with the definition of an open set in  $\mathbb{R}^2$ .

A set  $\Omega$  is **closed** if its complement  $\Omega^c = \mathbb{C} - \Omega$  is open. This property can be reformulated in terms of limit points. A point  $z \in \mathbb{C}$  is said to be a **limit point** of the set  $\Omega$  if there exists a sequence of points  $z_n \in \Omega$  such that  $z_n \neq z$  and  $\lim_{n \rightarrow \infty} z_n = z$ . The reader can now check that a set is closed if and only if it contains all its limit points. The **closure** of any set  $\Omega$  is the union of  $\Omega$  and its limit points, and is often denoted by  $\bar{\Omega}$ .

Finally, the **boundary** of a set  $\Omega$  is equal to its closure minus its interior, and is often denoted by  $\partial\Omega$ .

A set  $\Omega$  is **bounded** if there exists  $M > 0$  such that  $|z| < M$  whenever  $z \in \Omega$ . In other words, the set  $\Omega$  is contained in some large disc. If  $\Omega$  is bounded, we define its **diameter** by

$$\text{diam}(\Omega) = \sup_{z, w \in \Omega} |z - w|.$$

A set  $\Omega$  is said to be **compact** if it is closed and bounded. Arguing as in the case of real variables, one can prove the following.

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**Theorem 1.2** *The set  $\Omega \subset \mathbb{C}$  is compact if and only if every sequence  $\{z_n\} \subset \Omega$  has a subsequence that converges to a point in  $\Omega$ .*

An **open covering** of  $\Omega$  is a family of open sets  $\{U_\alpha\}$  (not necessarily countable) such that

$$\Omega \subset \bigcup_{\alpha} U_{\alpha}.$$

In analogy with the situation in  $\mathbb{R}$ , we have the following equivalent formulation of compactness.

**Theorem 1.3** *A set  $\Omega$  is compact if and only if every open covering of  $\Omega$  has a finite subcovering.*

Another interesting property of compactness is that of **nested sets**. We shall in fact use this result at the very beginning of our study of complex function theory, more precisely in the proof of Goursat's theorem in Chapter 2.

**Proposition 1.4** *If  $\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n \supset \cdots$  is a sequence of non-empty compact sets in  $\mathbb{C}$  with the property that*

$$\text{diam}(\Omega_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

*then there exists a unique point  $w \in \mathbb{C}$  such that  $w \in \Omega_n$  for all  $n$ .*

*Proof.* Choose a point  $z_n$  in each  $\Omega_n$ . The condition  $\text{diam}(\Omega_n) \rightarrow 0$  says precisely that  $\{z_n\}$  is a Cauchy sequence, therefore this sequence converges to a limit that we call  $w$ . Since each set  $\Omega_n$  is compact we must have  $w \in \Omega_n$  for all  $n$ . Finally,  $w$  is the unique point satisfying this property, for otherwise, if  $w'$  satisfied the same property with  $w' \neq w$  we would have  $|w - w'| > 0$  and the condition  $\text{diam}(\Omega_n) \rightarrow 0$  would be violated.

The last notion we need is that of **connectedness**. An open set  $\Omega \subset \mathbb{C}$  is said to be **connected** if it is not possible to find two disjoint non-empty open sets  $\Omega_1$  and  $\Omega_2$  such that

$$\Omega = \Omega_1 \cup \Omega_2.$$

A connected open set in  $\mathbb{C}$  will be called a **region**. Similarly, a closed set  $F$  is connected if one cannot write  $F = F_1 \cup F_2$  where  $F_1$  and  $F_2$  are disjoint non-empty closed sets.

There is an equivalent definition of connectedness for open sets in terms of curves, which is often useful in practice: an open set  $\Omega$  is connected if and only if any two points in  $\Omega$  can be joined by a curve  $\gamma$  entirely contained in  $\Omega$ . See Exercise 5 for more details.



## 2 Functions on the complex plane

### 2.1 Continuous functions

Let  $f$  be a function defined on a set  $\Omega$  of complex numbers. We say that  $f$  is **continuous** at the point  $z_0 \in \Omega$  if for every  $\epsilon > 0$  there exists  $\delta > 0$  such that whenever  $z \in \Omega$  and  $|z - z_0| < \delta$  then  $|f(z) - f(z_0)| < \epsilon$ . An equivalent definition is that for every sequence  $\{z_1, z_2, \dots\} \subset \Omega$  such that  $\lim z_n = z_0$ , then  $\lim f(z_n) = f(z_0)$ .

The function  $f$  is said to be continuous on  $\Omega$  if it is continuous at every point of  $\Omega$ . Sums and products of continuous functions are also continuous.

Since the notions of convergence for complex numbers and points in  $\mathbb{R}^2$  are the same, the function  $f$  of the complex argument  $z = x + iy$  is continuous if and only if it is continuous viewed as a function of the two real variables  $x$  and  $y$ .

By the triangle inequality, it is immediate that if  $f$  is continuous, then the real-valued function defined by  $z \mapsto |f(z)|$  is continuous. We say that  $f$  attains a **maximum** at the point  $z_0 \in \Omega$  if

$$|f(z)| \leq |f(z_0)| \quad \text{for all } z \in \Omega,$$

with the inequality reversed for the definition of a **minimum**.

**Theorem 2.1** *A continuous function on a compact set  $\Omega$  is bounded and attains a maximum and minimum on  $\Omega$ .*

This is of course analogous to the situation of functions of a real variable, and we shall not repeat the simple proof here.

### 2.2 Holomorphic functions

We now present a notion that is central to complex analysis, and in distinction to our previous discussion we introduce a definition that is genuinely *complex* in nature.

Let  $\Omega$  be an open set in  $\mathbb{C}$  and  $f$  a complex-valued function on  $\Omega$ . The function  $f$  is **holomorphic** at the point  $z_0 \in \Omega$  if the quotient

$$(1) \quad \frac{f(z_0 + h) - f(z_0)}{h}$$

converges to a limit when  $h \rightarrow 0$ . Here  $h \in \mathbb{C}$  and  $h \neq 0$  with  $z_0 + h \in \Omega$ , so that the quotient is well defined. The limit of the quotient, when it exists, is denoted by  $f'(z_0)$ , and is called the **derivative** of  $f$  at  $z_0$ :

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

It should be emphasized that in the above limit,  $h$  is a complex number that may approach 0 from any direction.

The function  $f$  is said to be **holomorphic** on  $\Omega$  if  $f$  is holomorphic at every point of  $\Omega$ . If  $C$  is a closed subset of  $\mathbb{C}$ , we say that  $f$  is **holomorphic on  $C$**  if  $f$  is holomorphic in some open set containing  $C$ . Finally, if  $f$  is holomorphic in all of  $\mathbb{C}$  we say that  $f$  is **entire**.

Sometimes the terms **regular** or **complex differentiable** are used instead of holomorphic. The latter is natural in view of (1) which mimics the usual definition of the derivative of a function of one real variable. But despite this resemblance, a holomorphic function of one complex variable will satisfy much stronger properties than a differentiable function of one real variable. For example, a holomorphic function will actually be infinitely many times complex differentiable, that is, the existence of the first derivative will guarantee the existence of derivatives of any order. This is in contrast with functions of one real variable, since there are differentiable functions that do not have two derivatives. In fact more is true: every holomorphic function is analytic, in the sense that it has a power series expansion near every point (power series will be discussed in the next section), and for this reason we also use the term **analytic** as a synonym for holomorphic. Again, this is in contrast with the fact that there are indefinitely differentiable functions of one real variable that cannot be expanded in a power series. (See Exercise 23.)

**EXAMPLE 1.** The function  $f(z) = z$  is holomorphic on any open set in  $\mathbb{C}$ , and  $f'(z) = 1$ . In fact, any polynomial

$$p(z) = a_0 + a_1 z + \dots + a_n z^n$$

is holomorphic in the entire complex plane and

$$p'(z) = a_1 + \dots + n a_n z^{n-1}.$$

This follows from Proposition 2.2 below.

**EXAMPLE 2.** The function  $1/z$  is holomorphic on any open set in  $\mathbb{C}$  that does not contain the origin, and  $f'(z) = -1/z^2$ .

**EXAMPLE 3.** The function  $f(z) = \bar{z}$  is not holomorphic. Indeed, we have

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\bar{h}}{h}$$

which has no limit as  $h \rightarrow 0$ , as one can see by first taking  $h$  real and then  $h$  purely imaginary.

An important family of examples of holomorphic functions, which we discuss later, are the power series. They contain functions such as  $e^z$ ,  $\sin z$ , or  $\cos z$ , and in fact power series play a crucial role in the theory of holomorphic functions, as we already mentioned in the last paragraph. Some other examples of holomorphic functions that will make their appearance in later chapters were given in the introduction to this book.

It is clear from (1) above that a function  $f$  is holomorphic at  $z_0 \in \Omega$  if and only if there exists a complex number  $a$  such that

$$(2) \quad f(z_0 + h) - f(z_0) - ah = h\psi(h),$$

where  $\psi$  is a function defined for all small  $h$  and  $\lim_{h \rightarrow 0} \psi(h) = 0$ . Of course, we have  $a = f'(z_0)$ . From this formulation, it is clear that  $f$  is continuous wherever it is holomorphic. Arguing as in the case of one real variable, using formulation (2) in the case of the chain rule (for example), one proves easily the following desirable properties of holomorphic functions.

**Proposition 2.2** *If  $f$  and  $g$  are holomorphic in  $\Omega$ , then:*

- (i)  $f + g$  is holomorphic in  $\Omega$  and  $(f + g)' = f' + g'$ .
- (ii)  $fg$  is holomorphic in  $\Omega$  and  $(fg)' = f'g + fg'$ .
- (iii) If  $g(z_0) \neq 0$ , then  $f/g$  is holomorphic at  $z_0$  and

$$(f/g)' = \frac{f'g - fg'}{g^2}.$$

Moreover, if  $f : \Omega \rightarrow U$  and  $g : U \rightarrow \mathbb{C}$  are holomorphic, the chain rule holds

$$(g \circ f)'(z) = g'(f(z))f'(z) \quad \text{for all } z \in \Omega.$$

### Complex-valued functions as mappings

We now clarify the relationship between the complex and real derivatives. In fact, the third example above should convince the reader that the notion of complex differentiability differs significantly from the usual notion of real differentiability of a function of two real variables. Indeed, in terms of real variables, the function  $f(z) = z$  corresponds to the map  $F : (x, y) \mapsto (x, -y)$ , which is differentiable in the real sense. Its derivative at a point is the linear map given by its Jacobian, the  $2 \times 2$  matrix of partial derivatives of the coordinate functions. In fact,  $F$  is linear and

is therefore equal to its derivative at every point. This implies that  $F$  is actually indefinitely differentiable. In particular the existence of the real derivative need not guarantee that  $f$  is holomorphic.

This example leads us to associate more generally to each complex-valued function  $f = u + iv$  the mapping  $F(x, y) = (u(x, y), v(x, y))$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ .

Recall that a function  $F(x, y) = (u(x, y), v(x, y))$  is said to be differentiable at a point  $P_0 = (x_0, y_0)$  if there exists a linear transformation  $J : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that

$$(3) \quad \frac{|F(P_0 + H) - F(P_0) - J(H)|}{|H|} \rightarrow 0 \quad \text{as } |H| \rightarrow 0, H \in \mathbb{R}^2.$$

Equivalently, we can write

$$F(P_0 + H) - F(P_0) = J(H) + |H|\Psi(H),$$

with  $|\Psi(H)| \rightarrow 0$  as  $|H| \rightarrow 0$ . The linear transformation  $J$  is unique and is called the derivative of  $F$  at  $P_0$ . If  $F$  is differentiable, the partial derivatives of  $u$  and  $v$  exist, and the linear transformation  $J$  is described in the standard basis of  $\mathbb{R}^2$  by the Jacobian matrix of  $F$

$$J = J_F(x, y) = \begin{pmatrix} \partial u / \partial x & \partial u / \partial y \\ \partial v / \partial x & \partial v / \partial y \end{pmatrix}.$$

In the case of complex differentiation the derivative is a complex number  $f'(z_0)$ , while in the case of real derivatives, it is a matrix. There is, however, a connection between these two notions, which is given in terms of special relations that are satisfied by the entries of the Jacobian matrix, that is, the partials of  $u$  and  $v$ . To find these relations, consider the limit in (1) when  $h$  is first real, say  $h = h_1 + ih_2$  with  $h_2 = 0$ . Then, if we write  $z = x + iy$ ,  $z_0 = x_0 + iy_0$ , and  $f(z) = f(x, y)$ , we find that

$$\begin{aligned} f'(z_0) &= \lim_{h_1 \rightarrow 0} \frac{f(x_0 + h_1, y_0) - f(x_0, y_0)}{h_1} \\ &= \frac{\partial f}{\partial x}(z_0), \end{aligned}$$

where  $\partial/\partial x$  denotes the usual partial derivative in the  $x$  variable. (We fix  $y_0$  and think of  $f$  as a complex-valued function of the single real variable  $x$ .) Now taking  $h$  purely imaginary, say  $h = ih_2$ , a similar argument yields



$$f'(z_0) = \lim_{h_2 \rightarrow 0} \frac{f(x_0, y_0 + h_2) - f(x_0, y_0)}{ih_2} = \frac{1}{i} \frac{\partial f}{\partial y}(z_0),$$

where  $\partial/\partial y$  is partial differentiation in the  $y$  variable. Therefore, if  $f$  is holomorphic we have shown that

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}.$$

Writing  $f = u + iv$ , we find after separating real and imaginary parts and using  $1/i = -i$ , that the partials of  $u$  and  $v$  exist, and they satisfy the following non-trivial relations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These are the **Cauchy-Riemann equations**, which link real and complex analysis.

We can clarify the situation further by defining two differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right).$$

**Proposition 2.3** *If  $f$  is holomorphic at  $z_0$ , then*

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \quad \text{and} \quad f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2 \frac{\partial u}{\partial z}(z_0).$$

Also, if we write  $F(x, y) = f(z)$ , then  $F$  is differentiable in the sense of real variables, and

$$\det J_F(x_0, y_0) = |f'(z_0)|^2.$$

*Proof.* Taking real and imaginary parts, it is easy to see that the Cauchy-Riemann equations are equivalent to  $\partial f/\partial \bar{z} = 0$ . Moreover, by our earlier observation

$$\begin{aligned} f'(z_0) &= \frac{1}{2} \left( \frac{\partial f}{\partial x}(z_0) + i \frac{\partial f}{\partial y}(z_0) \right) \\ &= \frac{\partial f}{\partial z}(z_0), \end{aligned}$$

and the Cauchy-Riemann equations give  $\partial f/\partial z = 2\partial u/\partial z$ . To prove that  $F$  is differentiable it suffices to observe that if  $H = (h_1, h_2)$  and  $h = h_1 + ih_2$ , then the Cauchy-Riemann equations imply

$$J_F(x_0, y_0)(H) = \left( \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) (h_1 + ih_2) = f'(z_0)h,$$

where we have identified a complex number with the pair of real and imaginary parts. After a final application of the Cauchy-Riemann equations, the above results imply that

$$(4) \quad \det J_F(x_0, y_0) = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} = \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 = \left| 2 \frac{\partial u}{\partial z} \right|^2 = |f'(z_0)|^2.$$

So far, we have assumed that  $f$  is holomorphic and deduced relations satisfied by its real and imaginary parts. The next theorem contains an important converse, which completes the circle of ideas presented here.

**Theorem 2.4** *Suppose  $f = u + iv$  is a complex-valued function defined on an open set  $\Omega$ . If  $u$  and  $v$  are continuously differentiable and satisfy the Cauchy-Riemann equations on  $\Omega$ , then  $f$  is holomorphic on  $\Omega$  and  $f'(z) = \partial f/\partial z$ .*

*Proof.* Write

$$u(x + h_1, y + h_2) - u(x, y) = \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 + |h| \psi_1(h)$$

and

$$v(x + h_1, y + h_2) - v(x, y) = \frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2 + |h| \psi_2(h),$$

where  $\psi_j(h) \rightarrow 0$  (for  $j = 1, 2$ ) as  $|h|$  tends to 0, and  $h = h_1 + ih_2$ . Using the Cauchy-Riemann equations we find that

$$f(z + h) - f(z) = \left( \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) (h_1 + ih_2) + |h| \psi(h),$$

where  $\psi(h) = \psi_1(h) + i\psi_2(h) \rightarrow 0$ , as  $|h| \rightarrow 0$ . Therefore  $f$  is holomorphic and

$$f'(z) = 2 \frac{\partial u}{\partial z} = \frac{\partial f}{\partial z}.$$

### 2.3 Power series

The prime example of a power series is the complex **exponential function**, which is defined for  $z \in \mathbb{C}$  by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

When  $z$  is real, this definition coincides with the usual exponential function, and in fact, the series above converges absolutely for every  $z \in \mathbb{C}$ . To see this, note that

$$\left| \frac{z^n}{n!} \right| = \frac{|z|^n}{n!},$$

so  $|e^z|$  can be compared to the series  $\sum |z|^n/n! = e^{|z|} < \infty$ . In fact, this estimate shows that the series defining  $e^z$  is uniformly convergent in every disc in  $\mathbb{C}$ .

In this section we will prove that  $e^z$  is holomorphic in all of  $\mathbb{C}$  (it is entire), and that its derivative can be found by differentiating the series term by term. Hence

$$(e^z)' = \sum_{n=0}^{\infty} n \frac{z^{n-1}}{n!} = \sum_{m=0}^{\infty} \frac{z^m}{m!} = e^z,$$

and therefore  $e^z$  is its own derivative.

In contrast, the geometric series

$$\sum_{n=0}^{\infty} z^n$$

converges absolutely only in the disc  $|z| < 1$ , and its sum there is the function  $1/(1-z)$ , which is holomorphic in the open set  $\mathbb{C} - \{1\}$ . This identity is proved exactly as when  $z$  is real: we first observe

$$\sum_{n=0}^N z^n = \frac{1 - z^{N+1}}{1 - z},$$

and then note that if  $|z| < 1$  we must have  $\lim_{N \rightarrow \infty} z^{N+1} = 0$ .

In general, a **power series** is an expansion of the form

$$(5) \quad \sum_{n=0}^{\infty} a_n z^n,$$

### 2. Functions on the complex plane

where  $a_n \in \mathbb{C}$ . To test for absolute convergence of this series, we must investigate

$$\sum_{n=0}^{\infty} |a_n| |z|^n,$$

and we observe that if the series (5) converges absolutely for some  $z_0$ , then it will also converge for all  $z$  in the disc  $|z| \leq |z_0|$ . We now prove that there always exists an open disc (possibly empty) on which the power series converges absolutely.

**Theorem 2.5** Given a power series  $\sum_{n=0}^{\infty} a_n z^n$ , there exists  $0 \leq R \leq \infty$  such that:

- (i) If  $|z| < R$  the series converges absolutely.
- (ii) If  $|z| > R$  the series diverges.

Moreover, if we use the convention that  $1/0 = \infty$  and  $1/\infty = 0$ , then  $R$  is given by Hadamard's formula

$$1/R = \limsup |a_n|^{1/n}.$$

The number  $R$  is called the **radius of convergence** of the power series, and the region  $|z| < R$  the **disc of convergence**. In particular, we have  $R = \infty$  in the case of the exponential function, and  $R = 1$  for the geometric series.

*Proof.* Let  $L = 1/R$  where  $R$  is defined by the formula in the statement of the theorem, and suppose that  $L \neq 0, \infty$ . (These two easy cases are left as an exercise.) If  $|z| < R$ , choose  $\epsilon > 0$  so small that

$$(L + \epsilon)|z| = r < 1.$$

By the definition  $L$ , we have  $|a_n|^{1/n} \leq L + \epsilon$  for all large  $n$ , therefore

$$|a_n| |z|^n \leq \{(L + \epsilon)|z|\}^n = r^n.$$

Comparison with the geometric series  $\sum r^n$  shows that  $\sum a_n z^n$  converges.

If  $|z| > R$ , then a similar argument proves that there exists a sequence of terms in the series whose absolute value goes to infinity, hence the series diverges.

**Remark.** On the boundary of the disc of convergence,  $|z| = R$ , the situation is more delicate as one can have either convergence or divergence. (See Exercise 19.)



Further examples of power series that converge in the whole complex plane are given by the standard trigonometric functions; these are defined by

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad \text{and} \quad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

and they agree with the usual cosine and sine of a real argument whenever  $z \in \mathbb{R}$ . A simple calculation exhibits a connection between these two functions and the complex exponential, namely,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

These are called the **Euler formulas** for the cosine and sine functions.

Power series provide a very important class of analytic functions that are particularly simple to manipulate.

**Theorem 2.6** *The power series  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  defines a holomorphic function in its disc of convergence. The derivative of  $f$  is also a power series obtained by differentiating term by term the series for  $f$ , that is,*

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}.$$

Moreover,  $f'$  has the same radius of convergence as  $f$ .

*Proof.* The assertion about the radius of convergence of  $f'$  follows from Hadamard's formula. Indeed,  $\lim_{n \rightarrow \infty} n^{1/n} = 1$ , and therefore

$$\limsup |a_n|^{1/n} = \limsup |n a_n|^{1/n},$$

so that  $\sum a_n z^n$  and  $\sum n a_n z^n$  have the same radius of convergence, and hence so do  $\sum a_n z^n$  and  $\sum n a_n z^{n-1}$ .

To prove the first assertion, we must show that the series

$$g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

gives the derivative of  $f$ . For that, let  $R$  denote the radius of convergence of  $f$ , and suppose  $|z_0| < r < R$ . Write

$$f(z) = S_N(z) + E_N(z),$$

where

$$S_N(z) = \sum_{n=0}^N a_n z^n \quad \text{and} \quad E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n.$$

Then, if  $h$  is chosen so that  $|z_0 + h| < r$  we have

$$\begin{aligned} \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) &= \left( \frac{S_N(z_0 + h) - S_N(z_0)}{h} - S'_N(z_0) \right) \\ &\quad + (S'_N(z_0) - g(z_0)) + \left( \frac{E_N(z_0 + h) - E_N(z_0)}{h} \right). \end{aligned}$$

Since  $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$ , we see that

$$\left| \frac{E_N(z_0 + h) - E_N(z_0)}{h} \right| \leq \sum_{n=N+1}^{\infty} |a_n| \left| \frac{(z_0 + h)^n - z_0^n}{h} \right| \leq \sum_{n=N+1}^{\infty} |a_n| n r^{n-1},$$

where we have used the fact that  $|z_0| < r$  and  $|z_0 + h| < r$ . The expression on the right is the tail end of a convergent series, since  $g$  converges absolutely on  $|z| < R$ . Therefore, given  $\epsilon > 0$  we can find  $N_1$  so that  $N > N_1$  implies

$$\left| \frac{E_N(z_0 + h) - E_N(z_0)}{h} \right| < \epsilon.$$

Also, since  $\lim_{N \rightarrow \infty} S'_N(z_0) = g(z_0)$ , we can find  $N_2$  so that  $N > N_2$  implies

$$|S'_N(z_0) - g(z_0)| < \epsilon.$$

If we fix  $N$  so that both  $N > N_1$  and  $N > N_2$  hold, then we can find  $\delta > 0$  so that  $|h| < \delta$  implies

$$\left| \frac{S_N(z_0 + h) - S_N(z_0)}{h} - S'_N(z_0) \right| < \epsilon,$$

simply because the derivative of a polynomial is obtained by differentiating it term by term. Therefore,

$$\left| \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) \right| < 3\epsilon$$

whenever  $|h| < \delta$ , thereby concluding the proof of the theorem.

Successive applications of this theorem yield the following.

**Corollary 2.7** *A power series is infinitely complex differentiable in its disc of convergence, and the higher derivatives are also power series obtained by termwise differentiation.*

We have so far dealt only with power series centered at the origin. More generally, a power series centered at  $z_0 \in \mathbb{C}$  is an expression of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

The disc of convergence of  $f$  is now centered at  $z_0$  and its radius is still given by Hadamard's formula. In fact, if

$$g(z) = \sum_{n=0}^{\infty} a_n z^n,$$

then  $f$  is simply obtained by translating  $g$ , namely  $f(z) = g(w)$  where  $w = z - z_0$ . As a consequence everything about  $g$  also holds for  $f$  after we make the appropriate translation. In particular, by the chain rule,

$$f'(z) = g'(w) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}.$$

A function  $f$  defined on an open set  $\Omega$  is said to be **analytic** (or have a **power series expansion**) at a point  $z_0 \in \Omega$  if there exists a power series  $\sum a_n (z - z_0)^n$  centered at  $z_0$ , with positive radius of convergence, such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{for all } z \text{ in a neighborhood of } z_0.$$

If  $f$  has a power series expansion at every point in  $\Omega$ , we say that  $f$  is **analytic on  $\Omega$** .

By Theorem 2.6, an analytic function on  $\Omega$  is also holomorphic there. A deep theorem which we prove in the next chapter says that the converse is true: every holomorphic function is analytic. For that reason, we use the terms holomorphic and analytic interchangeably.

### 3 Integration along curves

In the definition of a curve, we distinguish between the one-dimensional geometric object in the plane (endowed with an orientation), and its

parametrization, which is a mapping from a closed interval to  $\mathbb{C}$ , that is not uniquely determined.

A **parametrized curve** is a function  $z(t)$  which maps a closed interval  $[a, b] \subset \mathbb{R}$  to the complex plane. We shall impose regularity conditions on the parametrization which are always verified in the situations that occur in this book. We say that the parametrized curve is **smooth** if  $z'(t)$  exists and is continuous on  $[a, b]$ , and  $z'(t) \neq 0$  for  $t \in [a, b]$ . At the points  $t = a$  and  $t = b$ , the quantities  $z'(a)$  and  $z'(b)$  are interpreted as the one-sided limits

$$z'(a) = \lim_{h \rightarrow 0^+} \frac{z(a+h) - z(a)}{h} \quad \text{and} \quad z'(b) = \lim_{h \rightarrow 0^+} \frac{z(b+h) - z(b)}{h}.$$

In general, these quantities are called the right-hand derivative of  $z(t)$  at  $a$ , and the left-hand derivative of  $z(t)$  at  $b$ , respectively.

Similarly we say that the parametrized curve is **piecewise-smooth** if  $z$  is continuous on  $[a, b]$  and if there exist points

$$a = a_0 < a_1 < \cdots < a_n = b,$$

where  $z(t)$  is smooth in the intervals  $[a_k, a_{k+1}]$ . In particular, the right-hand derivative at  $a_k$  may differ from the left-hand derivative at  $a_k$  for  $k = 1, \dots, n-1$ .

Two parametrizations,

$$z : [a, b] \rightarrow \mathbb{C} \quad \text{and} \quad \tilde{z} : [c, d] \rightarrow \mathbb{C},$$

are **equivalent** if there exists a continuously differentiable bijection  $s \mapsto t(s)$  from  $[c, d]$  to  $[a, b]$  so that  $t'(s) > 0$  and

$$\tilde{z}(s) = z(t(s)).$$

The condition  $t'(s) > 0$  says precisely that the orientation is preserved: as  $s$  travels from  $c$  to  $d$ , then  $t(s)$  travels from  $a$  to  $b$ . The family of all parametrizations that are equivalent to  $z(t)$  determines a **smooth curve**  $\gamma \subset \mathbb{C}$ , namely the image of  $[a, b]$  under  $z$  with the orientation given by  $z$  as  $t$  travels from  $a$  to  $b$ . We can define a curve  $\gamma^-$  obtained from the curve  $\gamma$  by reversing the orientation (so that  $\gamma$  and  $\gamma^-$  consist of the same points in the plane). As a particular parametrization for  $\gamma^-$  we can take  $z^- : [a, b] \rightarrow \mathbb{R}^2$  defined by

$$z^-(t) = z(b + a - t).$$



It is also clear how to define a **piecewise-smooth curve**. The points  $z(a)$  and  $z(b)$  are called the **end-points** of the curve and are independent on the parametrization. Since  $\gamma$  carries an orientation, it is natural to say that  $\gamma$  begins at  $z(a)$  and ends at  $z(b)$ .

A smooth or piecewise-smooth curve is **closed** if  $z(a) = z(b)$  for any of its parametrizations. Finally, a smooth or piecewise-smooth curve is **simple** if it is not self-intersecting, that is,  $z(t) \neq z(s)$  unless  $s = t$ . Of course, if the curve is closed to begin with, then we say that it is simple whenever  $z(t) \neq z(s)$  unless  $s = t$ , or  $s = a$  and  $t = b$ .

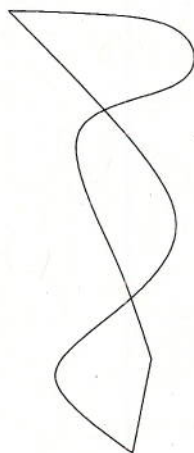


Figure 3. A closed piecewise-smooth curve

For brevity, we shall call any piecewise-smooth curve a **curve**, since these will be the objects we shall be primarily concerned with.

A basic example consists of a circle. Consider the circle  $C_r(z_0)$  centered at  $z_0$  and of radius  $r$ , which by definition is the set

$$C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}.$$

The **positive orientation** (counterclockwise) is the one that is given by the standard parametrization

$$z(t) = z_0 + re^{it}, \quad \text{where } t \in [0, 2\pi],$$

while the **negative orientation** (clockwise) is given by

$$z(t) = z_0 + re^{-it}, \quad \text{where } t \in [0, 2\pi].$$

In the following chapters, we shall denote by  $C$  a general *positively oriented* circle.

An important tool in the study of holomorphic functions is integration of functions along curves. Loosely speaking, a key theorem in complex

analysis says that if a function is holomorphic in the interior of a closed curve  $\gamma$ , then

$$\int_{\gamma} f(z) dz = 0,$$

and we shall turn our attention to a version of this theorem (called Cauchy's theorem) in the next chapter. Here we content ourselves with the necessary definitions and properties of the integral.

Given a smooth curve  $\gamma$  in  $\mathbb{C}$  parametrized by  $z : [a, b] \rightarrow \mathbb{C}$ , and  $f$  a continuous function on  $\gamma$ , we define the **integral of  $f$  along  $\gamma$**  by

$$\int_{\gamma} f(z) dz = \int_a^b f(z(t))z'(t) dt.$$

In order for this definition to be meaningful, we must show that the right-hand integral is independent of the parametrization chosen for  $\gamma$ . Say that  $\tilde{z}$  is an equivalent parametrization as above. Then the change of variables formula and the chain rule imply that

$$\int_a^b f(z(t))z'(t) dt = \int_c^d f(z(t(s)))z'(t(s))t'(s) ds = \int_c^d f(\tilde{z}(s))\tilde{z}'(s) ds.$$

This proves that the integral of  $f$  over  $\gamma$  is well defined.

If  $\gamma$  is piecewise smooth, then the integral of  $f$  over  $\gamma$  is simply the sum of the integrals of  $f$  over the smooth parts of  $\gamma$ , so if  $z(t)$  is a piecewise-smooth parametrization as before, then

$$\int_{\gamma} f(z) dz = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t) dt.$$

By definition, the **length** of the smooth curve  $\gamma$  is

$$\text{length}(\gamma) = \int_a^b |z'(t)| dt.$$

Arguing as we just did, it is clear that this definition is also independent of the parametrization. Also, if  $\gamma$  is only piecewise-smooth, then its length is the sum of the lengths of its smooth parts.

**Proposition 3.1** *Integration of continuous functions over curves satisfies the following properties:*

(i) It is linear, that is, if  $\alpha, \beta \in \mathbb{C}$ , then

$$\int_{\gamma} (\alpha f(z) + \beta g(z)) dz = \alpha \int_{\gamma} f(z) dz + \beta \int_{\gamma} g(z) dz.$$

(ii) If  $\gamma^-$  is  $\gamma$  with the reverse orientation, then

$$\int_{\gamma^-} f(z) dz = - \int_{\gamma} f(z) dz.$$

(iii) One has the inequality

$$\left| \int_{\gamma} f(z) dz \right| \leq \sup_{z \in \gamma} |f(z)| \cdot \text{length}(\gamma).$$

*Proof.* The first property follows from the definition and the linearity of the Riemann integral. The second property is left as an exercise. For the third, note that

$$\left| \int_{\gamma} f(z) dz \right| \leq \sup_{t \in [a, b]} |f(z(t))| \int_a^b |z'(t)| dt \leq \sup_{z \in \gamma} |f(z)| \cdot \text{length}(\gamma)$$

as was to be shown.

As we have said, Cauchy's theorem states that for appropriate closed curves  $\gamma$  in an open set  $\Omega$  on which  $f$  is holomorphic, then

$$\int_{\gamma} f(z) dz = 0.$$

The existence of primitives gives a first manifestation of this phenomenon. Suppose  $f$  is a function on the open set  $\Omega$ . A **primitive** for  $f$  on  $\Omega$  is a function  $F$  that is holomorphic on  $\Omega$  and such that  $F'(z) = f(z)$  for all  $z \in \Omega$ .

**Theorem 3.2** If a continuous function  $f$  has a primitive  $F$  in  $\Omega$ , and  $\gamma$  is a curve in  $\Omega$  that begins at  $w_1$  and ends at  $w_2$ , then

$$\int_{\gamma} f(z) dz = F(w_2) - F(w_1).$$

*Proof.* If  $\gamma$  is smooth, the proof is a simple application of the chain rule and the fundamental theorem of calculus. Indeed, if  $z(t) : [a, b] \rightarrow \mathbb{C}$  is a parametrization for  $\gamma$ , then  $z(a) = w_1$  and  $z(b) = w_2$ , and we have

$$\begin{aligned} \int_{\gamma} f(z) dz &= \int_a^b f(z(t)) z'(t) dt \\ &= \int_a^b F'(z(t)) z'(t) dt \\ &= \int_a^b \frac{d}{dt} F(z(t)) dt \\ &= F(z(b)) - F(z(a)). \end{aligned}$$

If  $\gamma$  is only piecewise-smooth, then arguing as we just did, we obtain a telescopic sum, and we have

$$\begin{aligned} \int_{\gamma} f(z) dz &= \sum_{k=0}^{n-1} F(z(a_{k+1})) - F(z(a_k)) \\ &= F(z(a_n)) - F(z(a_0)) \\ &= F(z(b)) - F(z(a)). \end{aligned}$$

**Corollary 3.3** If  $\gamma$  is a closed curve in an open set  $\Omega$ , and  $f$  is continuous and has a primitive in  $\Omega$ , then

$$\int_{\gamma} f(z) dz = 0.$$

This is immediate since the end-points of a closed curve coincide.

For example, the function  $f(z) = 1/z$  does not have a primitive in the open set  $\mathbb{C} - \{0\}$ , since if  $C$  is the unit circle parametrized by  $z(t) = e^{it}$ ,  $0 \leq t \leq 2\pi$ , we have

$$\int_C f(z) dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt = 2\pi i \neq 0.$$

In subsequent chapters, we shall see that this innocent calculation, which provides an example of a function  $f$  and closed curve  $\gamma$  for which  $\int_{\gamma} f(z) dz \neq 0$ , lies at the heart of the theory.

**Corollary 3.4** If  $f$  is holomorphic in a region  $\Omega$  and  $f' = 0$ , then  $f$  is constant.



*Proof.* Fix a point  $w_0 \in \Omega$ . It suffices to show that  $f(w) = f(w_0)$  for all  $w \in \Omega$ .

Since  $\Omega$  is connected, for any  $w \in \Omega$ , there exists a curve  $\gamma$  which joins  $w_0$  to  $w$ . Since  $f$  is clearly a primitive for  $f'$ , we have

$$\int_{\gamma} f'(z) dz = f(w) - f(w_0).$$

By assumption,  $f' = 0$  so the integral on the left is 0, and we conclude that  $f(w) = f(w_0)$  as desired.

**Remark on notation.** When convenient, we follow the practice of using the notation  $f(z) = O(g(z))$  to mean that there is a constant  $C > 0$  such that  $|f(z)| \leq C|g(z)|$  for  $z$  in a neighborhood of the point in question. In addition, we say  $f(z) = o(g(z))$  when  $|f(z)/g(z)| \rightarrow 0$ . We also write  $f(z) \sim g(z)$  to mean that  $f(z)/g(z) \rightarrow 1$ .

#### 4 Exercises

1. Describe geometrically the sets of points  $z$  in the complex plane defined by the following relations:

- (a)  $|z - z_1| = |z - z_2|$  where  $z_1, z_2 \in \mathbb{C}$ .
- (b)  $1/z = \bar{z}$ .
- (c)  $\operatorname{Re}(z) = 3$ .
- (d)  $\operatorname{Re}(z) > c$ , (resp.,  $\geq c$ ) where  $c \in \mathbb{R}$ .
- (e)  $\operatorname{Re}(az + b) > 0$  where  $a, b \in \mathbb{C}$ .
- (f)  $|z| = \operatorname{Re}(z) + 1$ .
- (g)  $\operatorname{Im}(z) = c$  with  $c \in \mathbb{R}$ .

2. Let  $\langle \cdot, \cdot \rangle$  denote the usual inner product in  $\mathbb{R}^2$ . In other words, if  $Z = (x_1, y_1)$  and  $W = (x_2, y_2)$ , then

$$\langle Z, W \rangle = x_1x_2 + y_1y_2.$$

Similarly, we may define a Hermitian inner product  $\langle \cdot, \cdot \rangle$  in  $\mathbb{C}$  by

$$\langle z, w \rangle = z\bar{w}.$$

#### 4. Exercises

The term Hermitian is used to describe the fact that  $\langle \cdot, \cdot \rangle$  is not symmetric, but rather satisfies the relation

$$\langle z, w \rangle = \overline{\langle w, z \rangle} \quad \text{for all } z, w \in \mathbb{C}.$$

Show that

$$\langle z, w \rangle = \frac{1}{2}[(z, w) + (w, z)] = \operatorname{Re}(z, w),$$

where we use the usual identification  $z = x + iy \in \mathbb{C}$  with  $(x, y) \in \mathbb{R}^2$ .

3. With  $\omega = se^{iv}$ , where  $s \geq 0$  and  $\varphi \in \mathbb{R}$ , solve the equation  $z^n = \omega$  in  $\mathbb{C}$  where  $n$  is a natural number. How many solutions are there?

4. Show that it is impossible to define a total ordering on  $\mathbb{C}$ . In other words, one cannot find a relation  $\succ$  between complex numbers so that:

- (i) For any two complex numbers  $z, w$ , one and only one of the following is true:  $z \succ w$ ,  $w \succ z$  or  $z = w$ .
- (ii) For all  $z_1, z_2, z_3 \in \mathbb{C}$  the relation  $z_1 \succ z_2$  implies  $z_1 + z_3 \succ z_2 + z_3$ .
- (iii) Moreover, for all  $z_1, z_2, z_3 \in \mathbb{C}$  with  $z_3 \succ 0$ , then  $z_1 \succ z_2$  implies  $z_1z_3 \succ z_2z_3$ .

[Hint: First check if  $i \succ 0$  is possible.]

5. A set  $\Omega$  is said to be **pathwise connected** if any two points in  $\Omega$  can be joined by a (piecewise-smooth) curve entirely contained in  $\Omega$ . The purpose of this exercise is to prove that an open set  $\Omega$  is pathwise connected if and only if  $\Omega$  is connected.

- (a) Suppose first that  $\Omega$  is open and pathwise connected, and that it can be written as  $\Omega = \Omega_1 \cup \Omega_2$  where  $\Omega_1$  and  $\Omega_2$  are disjoint non-empty open sets. Choose two points  $w_1 \in \Omega_1$  and  $w_2 \in \Omega_2$  and let  $\gamma$  denote a curve in  $\Omega$  joining  $w_1$  to  $w_2$ . Consider a parametrization  $z : [0, 1] \rightarrow \Omega$  of this curve with  $z(0) = w_1$  and  $z(1) = w_2$ , and let

$$t^* = \sup_{0 \leq s \leq 1} \{t : z(s) \in \Omega_1 \text{ for all } 0 \leq s < t\}.$$

Arrive at a contradiction by considering the point  $z(t^*)$ .

- (b) Conversely, suppose that  $\Omega$  is open and connected. Fix a point  $w \in \Omega$  and let  $\Omega_1 \subset \Omega$  denote the set of all points that can be joined to  $w$  by a curve contained in  $\Omega$ . Also, let  $\Omega_2 \subset \Omega$  denote the set of all points that cannot be joined to  $w$  by a curve in  $\Omega$ . Prove that both  $\Omega_1$  and  $\Omega_2$  are open, disjoint and their union is  $\Omega$ . Finally, since  $\Omega_1$  is non-empty (why?) conclude that  $\Omega = \Omega_1$ , as desired.

The proof actually shows that the regularity and type of curves we used to define pathwise connectedness can be relaxed without changing the equivalence between the two definitions when  $\Omega$  is open. For instance, we may take all curves to be continuous, or simply polygonal lines.<sup>2</sup>

6. Let  $\Omega$  be an open set in  $\mathbb{C}$  and  $z \in \Omega$ . The **connected component** (or simply the **component**) of  $\Omega$  containing  $z$  is the set  $C_z$  of all points  $w$  in  $\Omega$  that can be joined to  $z$  by a curve entirely contained in  $\Omega$ .

(a) Check first that  $C_z$  is open and connected. Then, show that  $w \in C_z$  defines an equivalence relation, that is: (i)  $z \in C_z$ , (ii)  $w \in C_z$  implies  $z \in C_w$ , and (iii) if  $w \in C_z$  and  $z \in C_z$ , then  $w \in C_z$ .

Thus  $\Omega$  is the union of all its connected components, and two components are either disjoint or coincide.

(b) Show that  $\Omega$  can have only countably many distinct connected components.

(c) Prove that if  $\Omega$  is the complement of a compact set, then  $\Omega$  has only one unbounded component.

[Hint: For (b), one would otherwise obtain an uncountable number of disjoint open balls. Now, each ball contains a point with rational coordinates. For (c), note that the complement of a large disc containing the compact set is connected.]

7. The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called **Blaschke factors**, will reappear in various applications in later chapters.

(a) Let  $z, w$  be two complex numbers such that  $\bar{z}w \neq 1$ . Prove that

$$\left| \frac{w-z}{1-\bar{w}z} \right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$\left| \frac{w-z}{1-\bar{w}z} \right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

[Hint: Why can one assume that  $z$  is real? It then suffices to prove that

$$(r-w)(r-\bar{w}) \leq (1-\tau w)(1-\tau\bar{w})$$

with equality for appropriate  $\tau$  and  $|w|$ .]

(b) Prove that for a fixed  $w$  in the unit disc  $\mathbb{D}$ , the mapping

$$F : z \mapsto \frac{w-z}{1-\bar{w}z}$$

satisfies the following conditions:

<sup>2</sup>A polygonal line is a piecewise-smooth curve which consists of finitely many straight line segments.

- (i)  $F$  maps the unit disc to itself (that is,  $F : \mathbb{D} \rightarrow \mathbb{D}$ ), and is holomorphic.  
 (ii)  $F$  interchanges 0 and  $w$ , namely  $F(0) = w$  and  $F(w) = 0$ .  
 (iii)  $|F(z)| = 1$  if  $|z| = 1$ .  
 (iv)  $F : \mathbb{D} \rightarrow \mathbb{D}$  is bijective. [Hint: Calculate  $F \circ F$ .]

8. Suppose  $U$  and  $V$  are open sets in the complex plane. Prove that if  $f : U \rightarrow V$  and  $g : V \rightarrow \mathbb{C}$  are two functions that are differentiable (in the real sense, that is, as functions of the two real variables  $x$  and  $y$ ), and  $h = g \circ f$ , then

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z} \frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{z}} \frac{\partial f}{\partial \bar{z}}$$

and

$$\frac{\partial h}{\partial \bar{z}} = \frac{\partial g}{\partial z} \frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{z}} \frac{\partial f}{\partial z}.$$

This is the complex version of the chain rule.

9. Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta \quad \text{where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

is holomorphic in the region  $r > 0$  and  $-\pi < \theta < \pi$ .

10. Show that

$$4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} = 4 \frac{\partial^2}{\partial z^2} - \Delta,$$

where  $\Delta$  is the Laplacian

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

11. Use Exercise 10 to prove that if  $f$  is holomorphic in the open set  $\Omega$ , then the real and imaginary parts of  $f$  are harmonic; that is, their Laplacian is zero.

12. Consider the function defined by

$$f(x+iy) = \sqrt{|x|}|y|, \quad \text{whenever } x, y \in \mathbb{R}.$$



Show that  $f$  satisfies the Cauchy-Riemann equations at the origin, yet  $f$  is not holomorphic at 0.

13. Suppose that  $f$  is holomorphic in an open set  $\Omega$ . Prove that in any one of the following cases:

- (a)  $\operatorname{Re}(f)$  is constant;
- (b)  $\operatorname{Im}(f)$  is constant;
- (c)  $|f|$  is constant;

one can conclude that  $f$  is constant.

14. Suppose  $\{a_n\}_{n=1}^N$  and  $\{b_n\}_{n=1}^N$  are two finite sequences of complex numbers. Let  $B_k = \sum_{n=1}^k b_n$  denote the partial sums of the series  $\sum b_n$  with the convention  $B_0 = 0$ . Prove the summation by parts formula

$$\sum_{n=M}^N a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n.$$

15. Abel's theorem. Suppose  $\sum_{n=1}^{\infty} a_n$  converges. Prove that

$$\lim_{r \rightarrow 1, r < 1} \sum_{n=1}^{\infty} r^n a_n = \sum_{n=1}^{\infty} a_n.$$

[Hint: Sum by parts.] In other words, if a series converges, then it is Abel summable with the same limit. For the precise definition of these terms, and more information on summability methods, we refer the reader to Book 1, Chapter 2.

16. Determine the radius of convergence of the series  $\sum_{n=1}^{\infty} a_n z^n$  when:

- (a)  $a_n = (\log n)^2$
- (b)  $a_n = n!$
- (c)  $a_n = \frac{n^2}{4^n + 3^n}$
- (d)  $a_n = (n!)^3 / (3n)!$  [Hint: Use Stirling's formula, which says that  $n! \sim cn^{n+1/2} e^{-n}$  for some  $c > 0$ .]

(e) Find the radius of convergence of the hypergeometric series

$$F(\alpha, \beta, \gamma; z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1) \cdots (\alpha+n-1) \beta(\beta+1) \cdots (\beta+n-1)}{n! \gamma(\gamma+1) \cdots (\gamma+n-1)} z^n.$$

Here  $\alpha, \beta \in \mathbb{C}$  and  $\gamma \neq 0, -1, -2, \dots$

(f) Find the radius of convergence of the Bessel function of order  $r$ :

$$J_r(z) = \left(\frac{z}{2}\right)^r \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{z}{2}\right)^{2n},$$

where  $r$  is a positive integer.

17. Show that if  $\{a_n\}_{n=0}^{\infty}$  is a sequence of non-zero complex numbers such that

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L,$$

then

$$\lim_{n \rightarrow \infty} |a_n|^{1/n} = L.$$

In particular, this exercise shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

18. Let  $f$  be a power series centered at the origin. Prove that  $f$  has a power series expansion around any point in its disc of convergence.

[Hint: Write  $z = z_0 + (z - z_0)$  and use the binomial expansion for  $z^n$ .]

19. Prove the following:

- (a) The power series  $\sum n z^n$  does not converge on any point of the unit circle.
- (b) The power series  $\sum z^n / n^2$  converges at every point of the unit circle.
- (c) The power series  $\sum z^n / n$  converges at every point of the unit circle except  $z = 1$ . [Hint: Sum by parts.]

20. Expand  $(1-z)^{-m}$  in powers of  $z$ . Here  $m$  is a fixed positive integer. Also, show that if

$$(1-z)^{-m} = \sum_{n=0}^{\infty} a_n z^n,$$

then one obtains the following asymptotic relation for the coefficients:

$$a_n \sim \frac{1}{(m-1)!} n^{m-1} \quad \text{as } n \rightarrow \infty.$$

21. Show that for  $|z| < 1$ , one has

$$\frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \cdots + \frac{z^{2^n}}{1-z^{2^{n+1}}} + \cdots = \frac{z}{1-z}.$$

and

$$\frac{z}{1+z} + \frac{2z^2}{1+z^2} + \cdots + \frac{2^k z^{2^k}}{1+z^{2^k}} + \cdots = \frac{z}{1-z}.$$

Justify any change in the order of summation.

[Hint: Use the dyadic expansion of an integer and the fact that  $2^{k+1} - 1 = 1 + 2 + 2^2 + \cdots + 2^k$ .]

22. Let  $N = \{1, 2, 3, \dots\}$  denote the set of positive integers. A subset  $S \subset N$  is said to be in arithmetic progression if

$$S = \{a, a+d, a+2d, a+3d, \dots\}$$

where  $a, d \in N$ . Here  $d$  is called the step of  $S$ .

Show that  $N$  cannot be partitioned into a finite number of subsets that are in arithmetic progression with distinct steps (except for the trivial case  $a = d = 1$ ).

[Hint: Write  $\sum_{n \in N} z^n$  as a sum of terms of the type  $\frac{z^a}{1-z^d}$ .]

23. Consider the function  $f$  defined on  $\mathbb{R}$  by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ e^{-1/x^2} & \text{if } x > 0. \end{cases}$$

Prove that  $f$  is indefinitely differentiable on  $\mathbb{R}$ , and that  $f^{(n)}(0) = 0$  for all  $n \geq 1$ . Conclude that  $f$  does not have a converging power series expansion  $\sum_{n=0}^{\infty} a_n x^n$  for  $x$  near the origin.

24. Let  $\gamma$  be a smooth curve in  $\mathbb{C}$  parametrized by  $z(t) : [a, b] \rightarrow \mathbb{C}$ . Let  $\gamma^-$  denote the curve with the same image as  $\gamma$  but with the reverse orientation. Prove that for any continuous function  $f$  on  $\gamma$

$$\int_{\gamma} f(z) dz = - \int_{\gamma^-} f(z) dz.$$

25. The next three calculations provide some insight into Cauchy's theorem, which we treat in the next chapter.

(a) Evaluate the integrals

$$\int_{\gamma} z^n dz$$

for all integers  $n$ . Here  $\gamma$  is any circle centered at the origin with the positive (counterclockwise) orientation.

(b) Same question as before, but with  $\gamma$  any circle not containing the origin.

(c) Show that if  $|a| < r < |b|$ , then

$$\int_{\gamma} \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b},$$

where  $\gamma$  denotes the circle centered at the origin, of radius  $r$ , with the positive orientation.

26. Suppose  $f$  is continuous in a region  $\Omega$ . Prove that any two primitives of  $f$  (if they exist) differ by a constant.