# MATH 105: REVIEW TO STOKES

Purpose is to review material of course and conclude with statement of at least one of Green/Gauss/Stokes Thm

↳ This is a massive generalization of the Fund Thm Calc

↳ Briefly, moving derivative from "function" to "region"

$$\iint \cdots \int_C d\omega = \int \cdots \int_{\partial C} \omega$$

where $\partial C$ is boundary of $C$

and $d\omega$ is derivative of $\omega$

## Sec 4.2: Arc Length


$c(t)$ : curve in 3-dim space (or n-dim space)

If $c(t) = (x(t), y(t), z(t))$

Then $c'(t) = (x'(t), y'(t), z'(t))$ is speed

and length of path $c(t)$ from $t_0$ to $t_1$ is

$$\int_{t_0}^{t_1} \|c'(t)\| \, dt = \int_{t_0}^{t_1} \sqrt{c'(t) \cdot c'(t)} \, dt$$

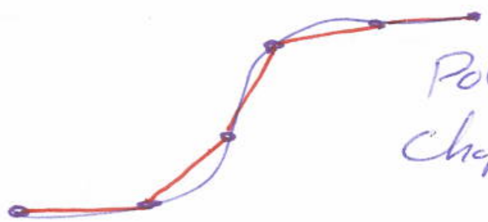
↳ most of the time these integrals are hard to do

↳ reduces to a Calc-II problem!

↳ comes down to parametrizing a curve

# Sec 4.2: Arc Length (cont)

Justification for formula



Polygonal approximation
Chop into many small time steps


length of red line segment is $\| C(t_{i+1}) - C(t_i) \|$

↳ using MVT lots of time, done

x-coord is $x(t_{i+1}) - x(t_i) = x'(s_i)\,\Delta t \approx x'(t_i)\,\Delta t$

$$y(t_{i+1}) - y(t_i) = y'(r_i)\,\Delta t \approx y'(t_i)\,\Delta t$$

$$z(t_{i+1}) - z(t_i) = z'(q_i)\,\Delta t \approx z'(t_i)\,\Delta t$$

yields red line segment has length $\approx \| C'(t_i) \| \, \Delta t$

So Length $\approx \sum \| C'(t_i) \| \, \Delta t \longrightarrow \int_{t=a}^{b} \| C'(t) \| \, dt$

<u>What does this review?</u>

↳ Parametrizing curves

↳ Riemann Sums

↳ MVT

HW: §4.2) #1 (See formula on bottom of page)

Suggested: #12, #18

# Section 7.1: The Path Integral

**Defn:** The path integral, or integral of $f$ along the path $c$ and denoted $\int_c f\,ds$ is $\int_a^b f(c(t))\,\|c'(t)\|\,dt$ where $c:[a,b]\to\mathbb{R}^3$ is a $\mathcal{C}^1$ map.

↳ idea: If $f=1$ get arc-length
Think of $f$ as a density

↳ See pg 424 for interpretation if $c:[a,b]\to\mathbb{R}^2$ ~~af~~ height $f(t)$ at $c(t)$
↳ gives area of fence of height $f(t)$ at $c(t)$

Ex: $f(x,y,z)=x+y+z$, $c(t)=(\sin t, \cos t, t)$ $0\le t\le 2\pi$

## What Does This Review

↳ Parametrizing curves

↳ Derivatives and integrals

Homework: #3b
Suggested: #6, #7, #13

# SECTION 7.2: LINE INTEGRALS

**Defn:** line integral of vector field $\vec{F}$ along a $C^1$ path $c$,

$c:[a,b] \to \mathbb{R}^3$, is

$$\int_c \vec{F}\cdot d\vec{s} = \int_a^b \vec{F}(c(t)) \cdot c'(t)\,dt$$

↳ interpretation: pages 429–430: work done

↳ if take $\vec{F}$ to be $c'(t)/\|c(t)\|$ (unit tangent)
then get arc length

**Notation:** $\vec{F} = (F_1, F_2, F_3)$, often write

$$\int_c \vec{F}\cdot d\vec{s} = \int_c F_1\,dx + \int_c F_2\,dy + \int_c F_3\,dz$$

which is $\displaystyle\int_c \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt$

$$= \int_c \vec{F}(c(t)) \cdot c'(t)\,dt$$

**Example:** $c(t) = (t, t^2, 1)$, $\vec{F}(x,y,z) = (x^2, xy, 1)$, $0 \le t \le 1$

So $c'(t) = (1, 2t, 0)$

$F(c(t)) = (t^2, t^3, 1)$

$$\int_c \vec{F}\cdot d\vec{s} = \int_0^1 (t^2 \cdot 1 + t^3 \cdot 2t + 1 \cdot 0)\,dt = \frac{11}{15}$$

$$\boxed{\text{KEY THM: } \int_c \nabla f \cdot d\vec{s} = f(c(b)) - f(c(a))}$$

only depends on the endpoints!

PROOF: If $F(t) = f(c(t))$ then $F'(t) = (\nabla f)(c(t)) \cdot c'(t)$

By FTC in 1-var: $\int_a^b F'(t)\,dt = F(b) - F(a)$ ∎

**What We Reviewed**: Chain Rule, Derivatives, FTC, paths

HW: #K

# Section 8.1: Green's Thm (AKA, Stoke's in the Plane)

**Green's Thm:** Let $D$ be a simple region with boundary $C$. Suppose $P, Q : D \to \mathbb{R}$ are $C^1$ functions. Then we have

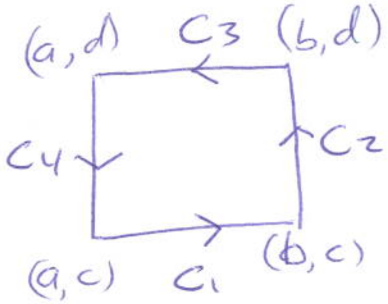
$$\int_{C^+} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where $C^+$ is the oriented boundary (st the region $D$ is on left as you travel).

Let $\vec{F} = (P, Q, 0)$, let $\partial D$ be boundary of $D$.

Rewrite: $$\int_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D (\nabla \times \vec{F}) \cdot \hat{k}\,dx\,dy$$

Proof: First do for a rectangle




Note on $C_1, C_3$ have $y'(t) = 0$

Note on $C_2, C_4$ have $x'(t) = 0$

$$\int_{C^+} P\,dx = \int_{C_1} P\,dx + \int_{C_3} P\,dx$$

$$= \int_a^b P(x,c)\,dx + \int_b^a P(x,d)\,dx$$

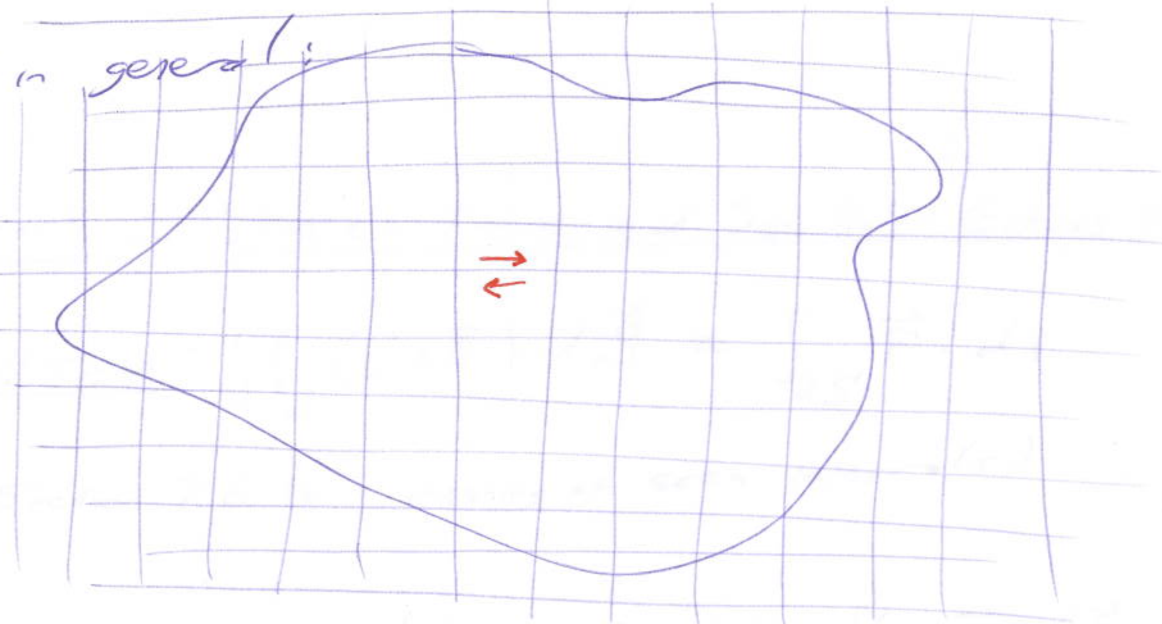
$$= \int_a^b \left[P(x,c) - P(x,d)\right] dx$$

$$= \int_a^b \int_c^d -\frac{\partial P}{\partial y}\,dy\,dx$$

similarly get $\int_{C^+} Q\,dy = \int_c^d \int_a^b \frac{\partial Q}{\partial x}\,dx\,dy$

# SECTION 8.1: GREEN'S THM (CONT)

Proof in general:



Interior line integrals cancel in pairs
left with line integral over boundary
↳ control errors

## What Did We Review?

↳ Double integrals
↳ Derivatives
↳ Cross Product / Partial Derivatives

Application: $\text{Area}(D) = \frac{1}{2} \iint_{\partial D} x\,dy - y\,dx$
↳ See example 2, page 524

See Thm 4, page 527

Homework: #3a

Suggested: #8