MATH 105: MULTIVARIABLE CALC

INTRO/OBJECTIVES

↳ Building block course

↳ Study how Things change/modeling

↳ Learn material and techniques!

↳ General math skills

- Good Notation
- Asking right question (motivation clip)
- Common techniques
- How to attack problems (brute force vs elegant)
- Building intuition/special cases

TYPES OF PROBLEMS

↳ Physics (forces, Newton's law)

↳ Economics (optimization)

↳ Finance (Monte-Carlo: dartboard)

↳ Geometry / Probability (areas/volumes)

↳ Approximation Theory (everything!)

↳ Discuss discovery of Neptune

↳ Library trip

COURSE MECHANICS

↳ Pace: faster than 104
learn by doing, daily HW

↳ Food / cell phones

↳ Read material before class

↳ 12 week vs 15 week
↳ supplemental lectures

↳ Office hours / on-line schedule
↳ strongly urge to visit
↳ TAs

THEME! CONTROL OVER YOUR EDUCATION

GRADING

↳ HW 15%

↳ Midterms 40% (2 or 3, best 1 or 2)

↳ Final 45%

↳ SCRIBE OPTION: REPLACE EXAMS: LOWER DECREASES BY 5%

↳ QUIZ OPTION: SAME; PROJECT: SAME

↳ CAN DO ~~BOTH~~ TWO FOR 10%, THREE FOR 15%

CHAPTER ONE: THE GEOMETRY OF EUCLIDEAN SPACE

GOAL: Set Notation
Review geometry and generalize
bank material for later.

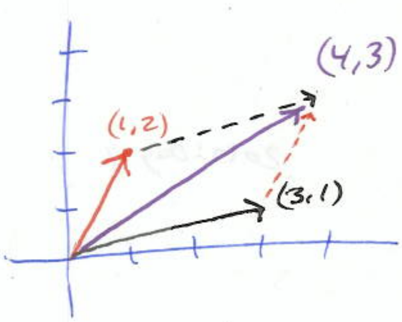
1.1. VECTORS IN 2 AND 3-DIM SPACES

All generalizes to n-dim space

↳ Motivate: paths of objects in physics
macro models

↳ Notation: $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ natural numbers
$\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ integers
$\mathbb{Q} = \{p/q,\ q \neq 0\}$ rationals
$\mathbb{R}$ = reals
$\mathbb{C}$ = Complex numbers: $x + iy$, $i = \sqrt{-1}$, $x, y \in \mathbb{R}$
$\mathbb{R}^n, \mathbb{C}^n$ = n-dim reals or complex

Cartesian Coordinates



Vectors: magnitude and direction

Add Componentwise

$\vec{x} = (x_1, \dots, x_n)$ $\vec{y} = (y_1, \dots, y_n)$

$\vec{x} + \vec{y} = (x_1 + y_1, \dots, x_n + y_n)$

$\alpha \vec{x} = (\alpha x_1, \dots, \alpha x_n)$

Section 1.1 (cont)

Basis:

Standard: $\vec{i}, \vec{j}, \vec{k}$ or $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \vec{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

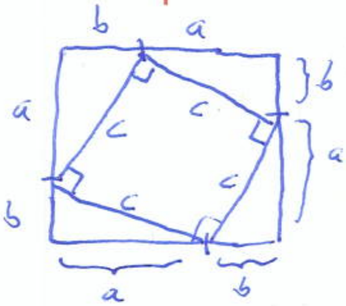
Sometimes use hat to indicate unit length: $\hat{i}, \hat{j}, \hat{k}$

↳ How do we measure length?

Pythagorean Thm

$$c^2 = a^2 + b^2$$

↳ So important we'll prove (Cleveland)



Thus $4 \cdot \frac{1}{2} ab + c^2 = (a+b)^2$

$$2ab + c^2 = a^2 + 2ab + b^2$$

$$\Rightarrow c^2 = a^2 + b^2 \qquad \blacksquare \text{ QED}$$

↳ Generalize to higher dimensions (Good Extra Credit: Prove!)

$\vec{v} = (x_1, \ldots, x_n)$

Then length of $\vec{v}$, denoted $\|\vec{v}\|$, is

$$\|\vec{v}\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Say unit length if $\|\vec{v}\| = 1$

(Proof in general use induction)

PROPERTIES OF VECTOR ADDITION

Everything you would expect : $\vec{x}, \vec{y}, \vec{z}$ vectors, α, β scalars

↳ assoc : $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$

comm : $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

distributive: $\alpha(\vec{x} + \vec{y}) = \alpha\vec{x} + \alpha\vec{y}$

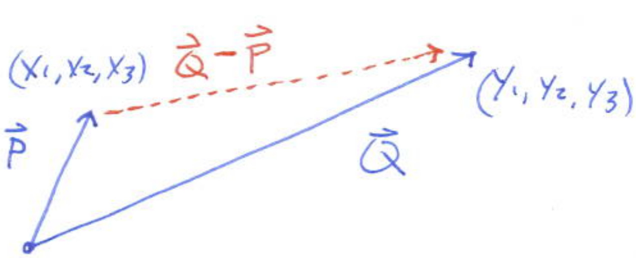
$(\alpha + \beta)\vec{x} = \alpha\vec{x} + \beta\vec{x}$

special: $0 \cdot \vec{x} = \vec{0}$, $1 \cdot \vec{x} = \vec{x}$

USING A BASIS

$$(3, 4, 1701) = 3\hat{i} + \cancel{8}\hat{j} + 1701\hat{k}$$
$$= 3\vec{i} + 4\vec{j} + 1701\vec{k}$$
$$= 3\vec{e}_1 + 4\vec{e}_2 + 1701\vec{e}_3$$

ADDING / SUBTRACTING VECTORS



$$\vec{Q} - \vec{P} = (y_1 - x_1, y_2 - x_2, y_3 - x_3)$$

Other notations: $\vec{P}_1 = (x_1, y_1, z_1)$

$\vec{P}_2 = (x_2, y_2, z_2)$

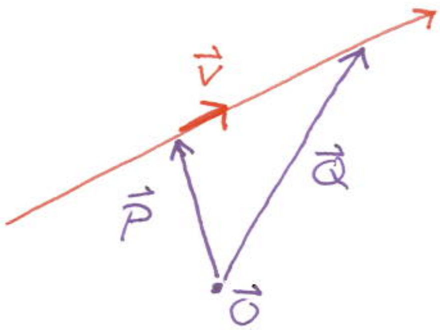
or Q is $\vec{P}' = (x_1', x_2', x_3')$ ...

EQUATION OF A LINE

Always start in one-dim and try to generalize

$y = mx + b \qquad \frac{y - y_1}{x - x_1} = m \iff y = y_1 + m(x - x_1)$

$y - y_1 = m(x - x_1)$



Want $\vec{Q} - \vec{P}$ to be a multiple of $\vec{v}$

Line is $\vec{Q} = \vec{P} + t\vec{v}$

Given point $\vec{P}$ on line ℓ with direction $\vec{v}$

↳ $\ell(t) = \vec{P} + t\vec{v}$

↳ if $\vec{P} = (P_1, \ldots, P_n)$ and $\vec{v} = (v_1, \ldots, v_n)$ Then

$x_1 = P_1 + t v_1$

$\vdots$

$x_n = P_n + t v_n$

In $\mathbb{R}^3$ often write $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} P_1 + t v_1 \\ P_2 + t v_2 \\ P_3 + t v_3 \end{pmatrix}$ or $\begin{pmatrix} x_1 + t v_1 \\ x_2 + t v_2 \\ x_3 + t v_3 \end{pmatrix}$

Given two points $\vec{P}, \vec{Q}$ on line, have

$\ell(t) = \vec{P} + (\vec{Q} - \vec{P})t$

Example:

$\vec{P} = (1,2,3)$ $\vec{v} = (0,1,-1)$

$\ell(t) = (1,2,3) + t(0,1,-1)$

$= (1, 2+t, 3-t)$

so $x = x(t) = 1$

$y = y(t) = 2+t$

$z = z(t) = 3-t$

$\vec{P} = (1,2,3)$ $\vec{Q} = (1,4,1)$

$\vec{Q} - \vec{P} = (1,4,1) - (1,2,3)$

$= (1-1, 4-2, 1-3)$

$= (0, 2, -2)$

$\ell(t) = \vec{P} + \cancel{t}s(\vec{Q} - \vec{P})$

$= (1,2,3) + \cancel{t}s(0,2,-2)$

$= 1, 2 + 2\cancel{t}s, 3 - 2s)$

so $x = x(s) = 1$

$y = y(s) = 2 + 2s$

$z = z(s) = 3 - 2s$

"look" different, but same line (send / replace s with $2t$)

↳ changes how fast travel

↳ could make look really different

↳ pattern recognition crucial

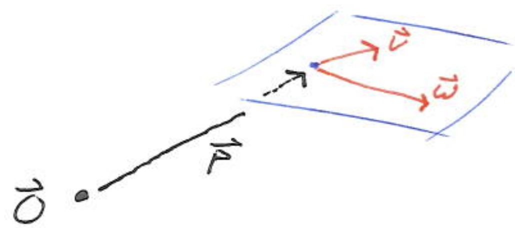
↳ telescoping sums example from FTC

Section 1.1 (cont)

Planes

Generalize what did for lines

Input: point and two directions



Plane is all Q satisfying $\vec{Q} - \vec{P} = t\vec{v} + s\vec{w}$ for some t, s

$$\text{Plane}(t,s) = \{\vec{Q} : \vec{Q} = \vec{P} + t\vec{v} + s\vec{w},\ t, s \in \mathbb{R}\}$$

Say plane is spanned by $\vec{v}$ and $\vec{w}$ (do more in lin alg)

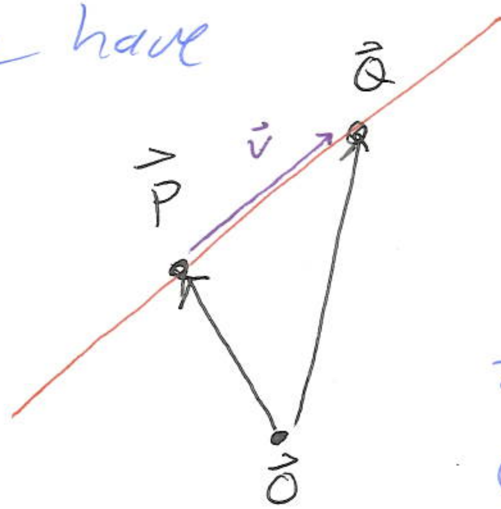
Ex: $\{(2,1,0) + t(1,0,-1) + s(2,4,6)\}$

HW: #4, #7, #13, #16, #22

Suggested: #9, #19, #28, #30 chemistry, in plane prod slopes $\perp$ lines is -1

MATH 105: LINES IN SPACES

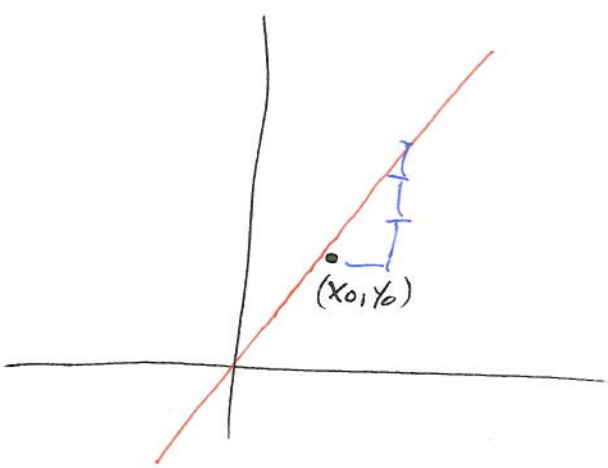
The following might help look at generalizing equations of lines. In three dimensional space we have



The direction of the line is $\vec{v} = \vec{Q} - \vec{P}$

The equation of the line is
$(x, y, z) = \vec{P} + t\vec{v}$

Some people have had some trouble seeing this as a generalization of the standard line in the plane, so I thought I'd provide another attempt at explaining it.



Here is a line going through the point (x_0, y_0) with slope 3. We may write this as
$y - y_0 = 3(x - x_0)$

Also Sec 1.1 Appendix: Page 8A

We note that this line is in the direction $(1,3)$; for every one unit of x we move, we go three units in the y-direction.

In our notation, we have the anchor point (x_0, y_0) with direction $(1,3)$ (which is like a slope of 3)

$$(x,y) = (x_0,y_0) + t(1,3)$$

$$(x,y) = (x_0 + t,\ y_0 + 3t)$$

Note: two equations: $x = x_0 + t$
$y = y_0 + 3t$

subtracting: $x - x_0 = t$
$y - y_0 = 3t$

Thus $y - y_0 = 3(x - x_0)$,
the old equation for the line!

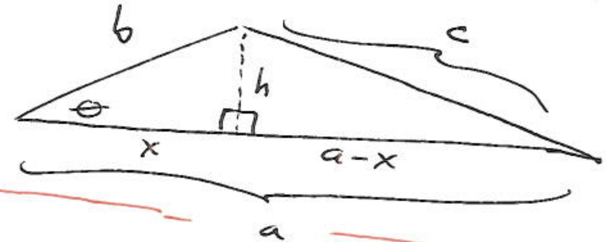
Also Sec 1.1 Appendix
Page 8B

SECTION 1.2: INNER PRODUCT, LENGTH AND DISTANCE

GOAL: understand angle between vectors

LAW OF COSINES:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$



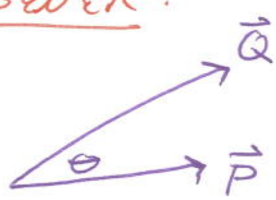
Proof: Drop auxiliary line h, two right triangles

$$h^2 = b^2 - x^2 = c^2 - (a-x)^2$$

Expanding $\Rightarrow$ $c^2 = a^2 + b^2 - 2ax$, but $x = b\cos\theta$ ∎

QUESTION: Given two vectors $\vec{P}$ and $\vec{Q}$, find angle b/w them in terms of coords

ANSWER:



$$\vec{P}\cdot\vec{Q} = \|\vec{P}\|\ \|\vec{Q}\| \cos\theta$$

with $\vec{P} = (p_1, \dots, p_n)$ $\vec{Q} = (q_1, \dots, q_n)$

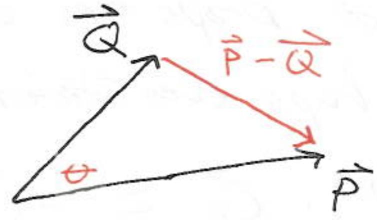
and $\vec{P}\cdot\vec{Q} = p_1q_1 + \dots + p_nq_n = \sum_{i=1}^{n} p_iq_i$

Call $\vec{P}\cdot\vec{Q}$ the dot product or the inner product

↳ EMPHASIZE GOOD FEATURES OF FORMULA

↳ $\vec{P} \to \alpha\vec{P}$, $\vec{Q} \to \beta\vec{Q}$ doesn't change angle

PROOF OF ANGLE FORMULA



$$\|\vec{P}\|^2 = p_1^2 + \cdots + p_n^2 = \vec{P}\cdot\vec{P}$$

$$\|\vec{Q}\|^2 = q_1^2 + \cdots + q_n^2 = \vec{Q}\cdot\vec{Q}$$

Law of Cosines: $\|\vec{P}-\vec{Q}\|^2 = \|\vec{P}\|^2 + \|\vec{Q}\|^2 - 2\|\vec{P}\|\,\|\vec{Q}\|\cos\theta$

so $\|(p_1-q_1,\dots,p_n-q_n)\|^2 = \|\vec{P}\|^2 + \|\vec{Q}\|^2 - 2\|\vec{P}\|\,\|\vec{Q}\|\cos\theta$

$$\sum_{i=1}^{n}(p_i-q_i)^2 = \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} q_i^2 - 2\|\vec{P}\|\,\|\vec{Q}\|\cos\theta$$

$$\sum_{i=1}^{n}(\cancel{p_i^2} - \cancel{2}p_iq_i + \cancel{q_i^2}) = \cancel{\sum_{i=1}^{n} p_i^2} + \cancel{\sum_{i=1}^{n} q_i^2} - \cancel{2}\|\vec{P}\|\,\|\vec{Q}\|\cos\theta$$

$$\Rightarrow \sum_{i=1}^{n} p_iq_i = \|\vec{P}\|\,\|\vec{Q}\|\cos\theta$$

or $\vec{P}\cdot\vec{Q} = \|\vec{P}\|\,\|\vec{Q}\|\cos\theta$ ∎

↳ "Faster" Proof

$$\|\vec{P}-\vec{Q}\|^2 = (\vec{P}-\vec{Q})\cdot(\vec{P}-\vec{Q}) = \|\vec{P}\|^2 - 2\,\vec{P}\cdot\vec{Q} + \|\vec{Q}\|^2$$

↳ Corollary: Cauchy-Schwarz Ineq

$$|\vec{P}\cdot\vec{Q}| \le \|\vec{P}\|\,\|\vec{Q}\|$$

SECTION 1.2 (CONT)

↳ TRIANGLE INEQ

$$\|\vec{P} + \vec{Q}\| \le \|\vec{P}\| + \|\vec{Q}\|$$

↳ Proof: Square both sides and compare
note $|\cos\theta| \le 1$

↳ ORTHOGONAL PROJECTIONS

Skip for now, revisit after linear algebra

↳ Physical Applications of Vectors

Physics examples (forces, ...)

Homework: #1, #7, #19

Suggested: #20, #27 (physics), Find a relation b/w the sum of the lengths of the two diagonals of a parallelogram and the sum of the lengths of the sides. Prove your relation.

SECTION 1.3: MATRICES, DETERMINANTS AND THE CROSS PRODUCT

Need linear algebra for $n \geq 4$, not too bad if $n \leq 3$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Then $\mathrm{Det}(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ Then $\mathrm{Det}(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - gec - hfa - idb$

↳ Trick: $\begin{array}{c} - \quad - \quad - \\ \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix} \\ + \quad + \quad + \end{array}$

copy first two columns
6 products to combine
↳ Three with + sign
Three with − sig

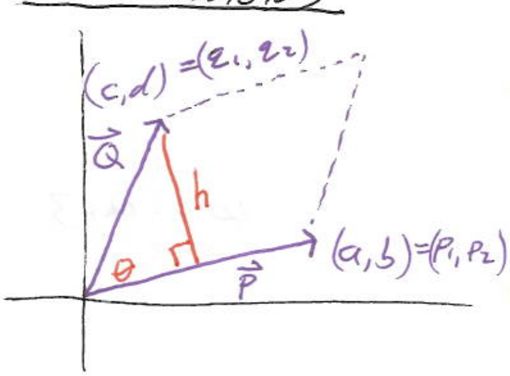
↳ **WARNING: ONLY WORKS FOR $n=3$!!**

GEOMETRIC INTERPRETATION

$A = \begin{pmatrix} \cdots \vec{v}_1 \cdots \\ \vdots \\ \cdots \vec{v}_n \cdots \end{pmatrix}$, determinant gives signed volume

for generalized parallelogram spanned by $\vec{v}_1, \dots, \vec{v}_n$.

↳ 2 DIMENSIONS



Area is base * height (prove if desired)

base is $\|\vec{P}\|$, $h = \|\vec{Q}\| \cancel{\cos}\ \sin$

now $\cos\theta = \vec{P} \cdot \vec{Q} / \|\vec{P}\| \, \|\vec{Q}\|$

so Area = ?

Section 1.3 (cont)

Geometric Interpretation (cont)

Area is $\|\vec{P}\| \, \|\vec{Q}\| \sin\theta$, $\sin\theta = (1-\cos^2\theta)^{1/2}$

$$\text{Area}^2 = \|\vec{P}\|^2 \|\vec{Q}\|^2 (1-\cos^2\theta)$$

$$= \|\vec{P}\|^2 \|\vec{Q}\|^2 - \|\vec{P}\|^2 \|\vec{Q}\|^2 \left(\frac{\vec{P}\cdot\vec{Q}}{\|\vec{P}\|\,\|\vec{Q}\|}\right)^2$$

$$= \|\vec{P}\|^2 \|\vec{Q}\|^2 - (\vec{P}\cdot\vec{Q})^2$$

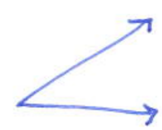
$$= (p_1^2+p_2^2)(q_1^2+q_2^2) - (p_1q_1+p_2q_2)^2$$

$\Big\downarrow$ algebra

$$= (p_1q_2 - p_2q_1)^2 \; !$$

unenlightening!

↳ Gain intuition from special cases:



Cross Product

Only in $\mathbb{R}^3$

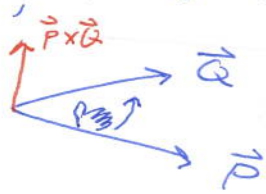
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix} = (p_2q_3 - p_3q_2)\hat{i} - (p_1q_3 + p_3q_1)\hat{j} + (p_1q_2 - p_2q_1)\hat{k}$$

$$= \vec{P} \times \vec{Q} \qquad = (p_2q_3 - p_3q_2,\ p_3q_1 - p_1q_3,\ p_1q_2 - p_2q_1)$$

PROPERTIES OF THE CROSS PRODUCT

- $\|\vec{P} \times \vec{Q}\| = \|\vec{P}\| \cdot \|\vec{Q}\| \cdot \sin\theta$ = area of parallelogram
- $\vec{P} \times \vec{P} = \vec{0}$
- $\vec{P} \times \vec{Q} = -\vec{Q} \times \vec{P}$
- $\vec{P} \times \vec{Q} = \vec{0}$ if and only if $\vec{P}$ and $\vec{Q}$ parallel
- $\alpha(\vec{P} \times \vec{Q}) = \alpha\vec{P} \times \vec{Q} = \vec{P} \times \alpha\vec{Q}$
- $\vec{P} \times (\vec{Q} + \vec{R}) = \vec{P} \times \vec{Q} + \vec{P} \times \vec{R}$
- $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$

  ↳ Right hand rule



EXTRA CREDIT: IS THE CROSS PRODUCT ASSOCIATIVE? DOES $\vec{P} \times (\vec{Q} \times \vec{R}) = (\vec{P} \times \vec{Q}) \times \vec{R}$? PROVE OR DISPROVE

- TRIPLE PRODUCT: $(\vec{A} \times \vec{B}) \cdot \vec{C}$ —algebra⟹ $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

- NOTE $\vec{P}$ and $\vec{Q}$ are perpendicular to $\vec{P} \times \vec{Q}$

Section 1.3 (cont)

Eq of Planes

Given Point $\vec{P}$ and normal direction $\vec{n}$, plane is all $\vec{Q}$ st $(\vec{Q} - \vec{P}) \cdot \vec{n} = 0$

~~Ex: $P = (P_1, P_2, P_3)$ and $\vec{n} = (n_1, n_2, n_3)$ and $\vec{Q} = (x, y, z)$~~

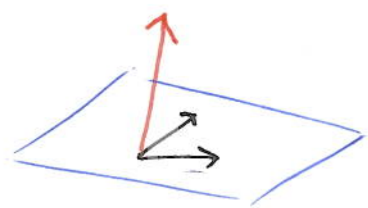
Ex: ~~Get~~ $\vec{P} = (x_0, y_0, z_0)$ $\vec{n} = (a, b, c)$ $\vec{Q} = (x, y, z)$

Equation is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

Notes: Read Historical Notes

↳ Can rescale normal: same equation

↳ if know two points in plane, can determine normal direction by the cross product!



↳ Special feature of $\mathbb{R}^3$

Only 3 dirs, know two then know the third!

Homework: #2c, #4, #6, #15a

Suggested: #10, #17a, #21, #35 (physics)

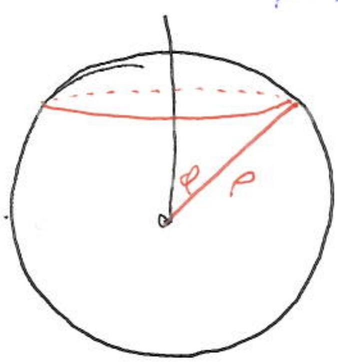
Section 1.4: Cylindrical + Spherical Coords

Polar: $x = r\cos\theta \quad y = r\sin\theta \quad \theta \in [0, 2\pi)$,

Cylindrical: $x = r\cos\theta \quad y = r\sin\theta \quad z = z$,

Spherical: $x = \rho\sin\varphi\cos\theta \quad y = \rho\sin\varphi\sin\theta \quad z = \rho\cos\varphi$

$\rho \geq 0 \quad \theta \in [0, 2\pi) \quad \varphi \in [0, \pi]$



Homework: #6, #8

Suggested: #14

Section 1.5: n-Dim Euclidean Space

Already did most of this

$\vec{x} = (x_1, \dots, x_n)$ then transpose is $\vec{x}^T = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$AB = \begin{pmatrix} a_{i1} \cdots a_{in} \end{pmatrix} \begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix} = \begin{pmatrix} c_{ij} \end{pmatrix}$$

with $c_{ij} = \vec{a}_{i\cdot} \cdot \vec{b}_{\cdot j}$

↳ typically $AB \neq BA$

Suggested HW: #19