CHAPTER TWO: DIFFERENTIATION

SECTION 2.1: GEOMETRY OF REAL VALUED FNS

Setup: $f: \mathbb{R}^n \to \mathbb{R}^m$

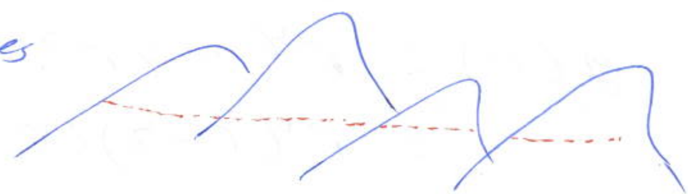
↳ vector valued if $m \geq 2$

scaler valued if $m = 1$

- $f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$: graph$(f) = \{(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) : \vec{x} \in U\}$
- Level set is subset where f is constant

↳ level set of value c is $\{\vec{x} \in U : f(\vec{x}) = c\}$

↳ Think heights in mountain ranges



Lots of examples/additional notation in the book

↳ Many graphing programs will do this

↳ Do Mathematica Example

Homework: #1, #24

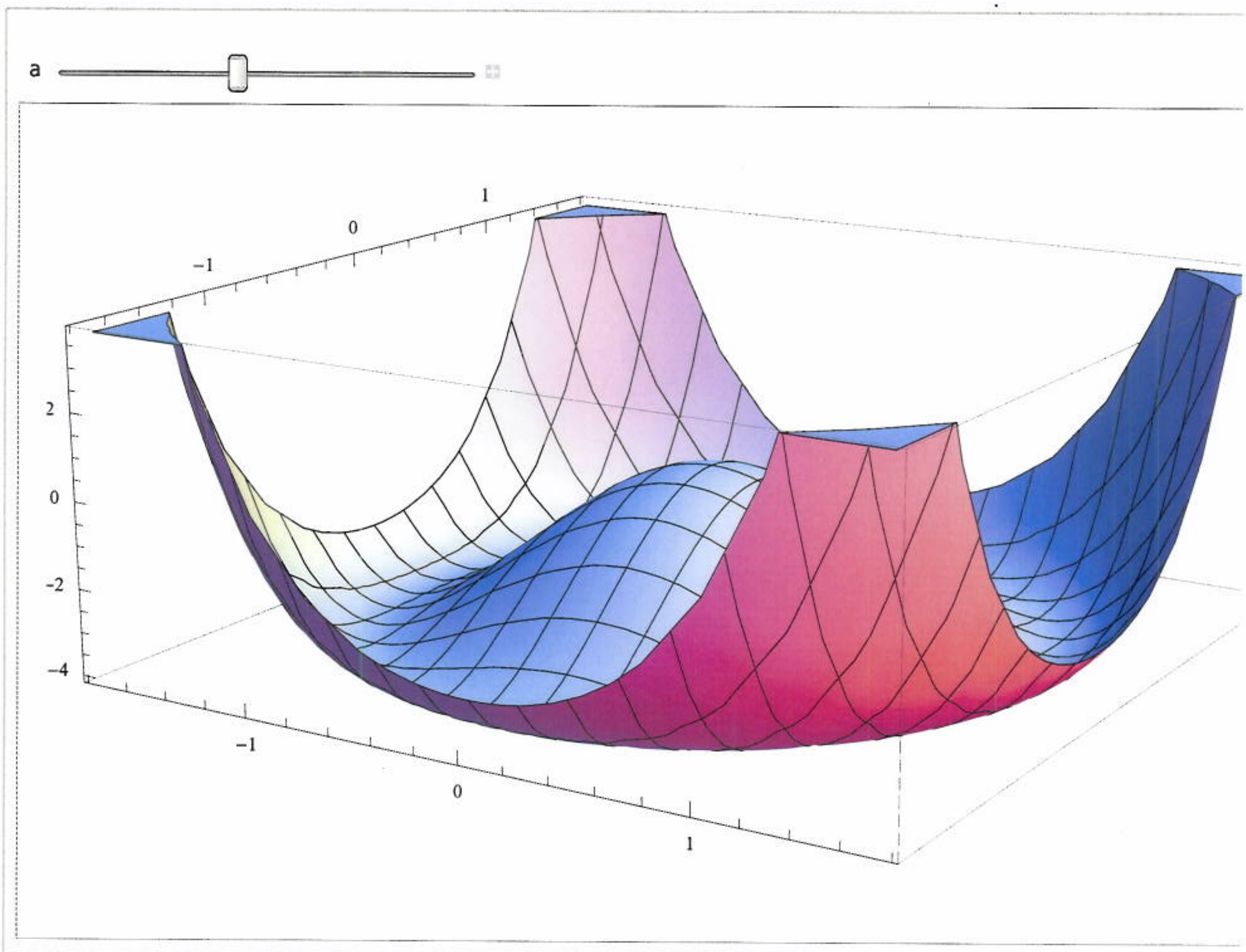
Suggested: #3, #30

Math 105 : Level Sets and Contour Plots

In[9]:=
```
f[x_, y_] := (x^2 + y^2)^2 - 4(x^2 + y^2)
```

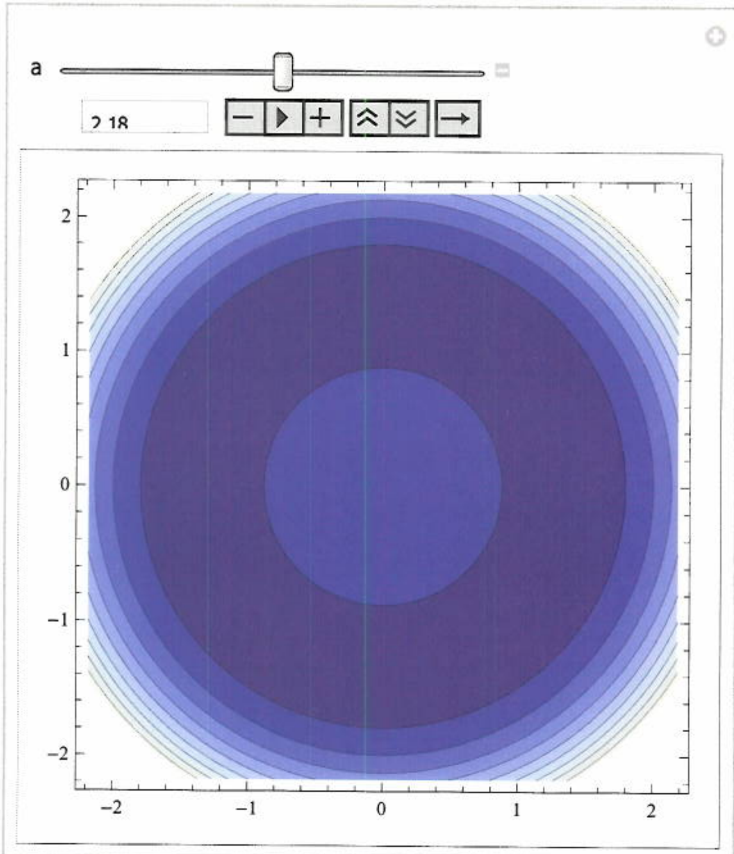
In[10]:=
```
Manipulate[
  Plot3D[f[x, y], {x, -a, a}, {y, -a, a}], {a, .1, 4}]
```

Out[10]=



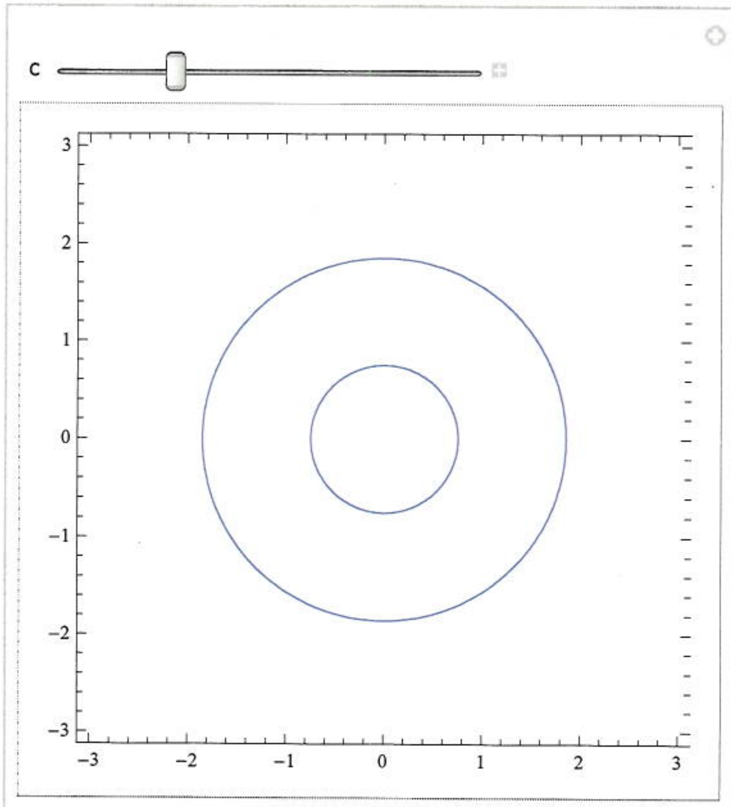
In[11]:=
```
Manipulate[ContourPlot[f[x, y],
    {x, -a, a}, {y, -a, a}], {a, .1, 4}]
```

Out[11]=



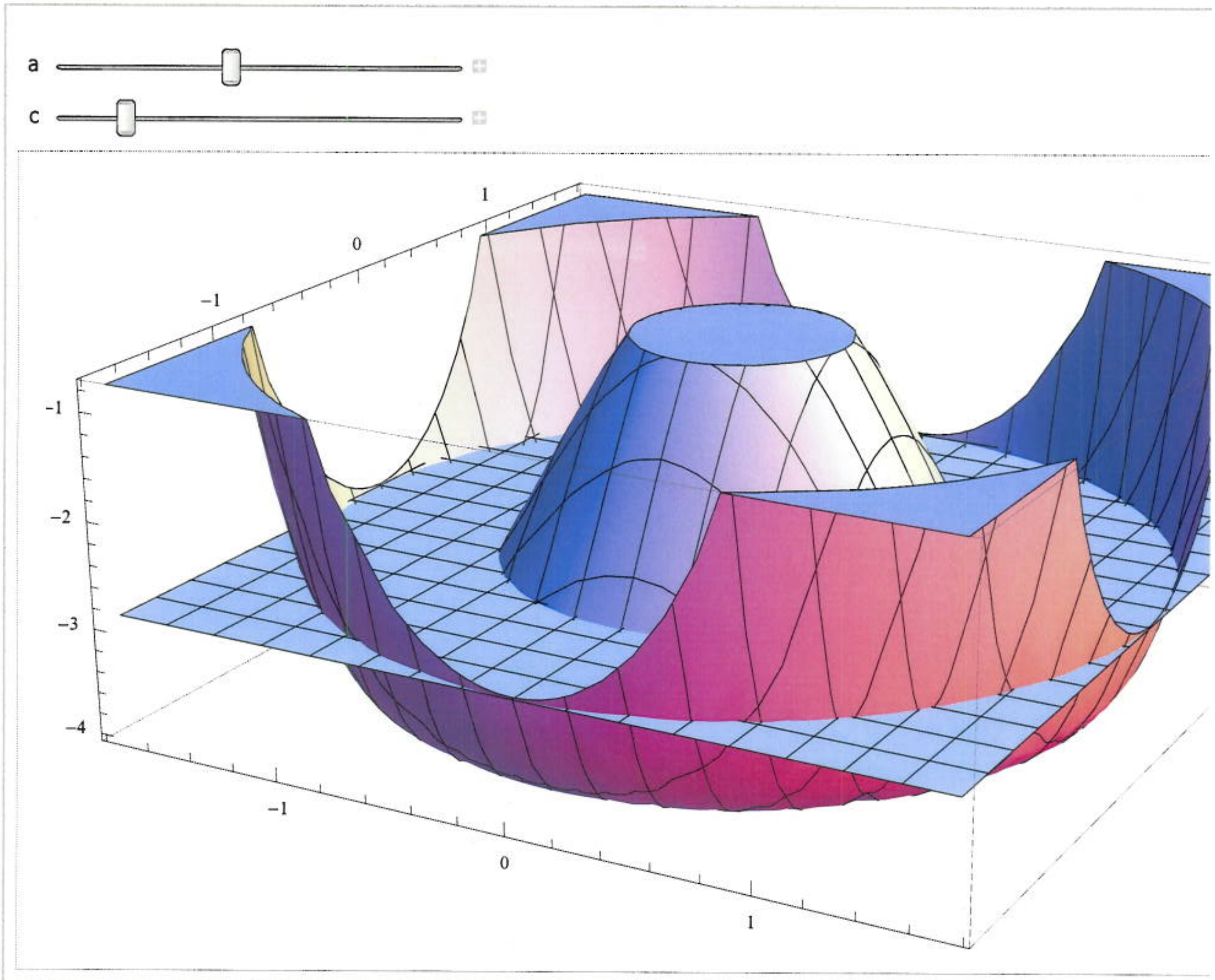
In[13]:=
```
Manipulate[ContourPlot[f[x, y] == c,
    {x, -3, 3}, {y, -3, 3}], {c, -4, 4}]
```

Out[13]=



In[14]:= **Manipulate[Plot3D[{f[x, y], c}, {x, −a, a}, {y, −a, a}], {a, .1, 4}, {c, −4, 4}]**

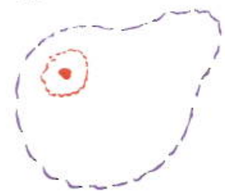
Out[14]=



SECTION 2.2: LIMITS AND CONTINUITY

TERMINOLOGY

- **OPEN DISK/BALL**: $D_r(\vec{x}_0) = \{\vec{x} : \|\vec{x} - \vec{x}_0\| < r\}$
- **OPEN SET**: $U \subset \mathbb{R}^n$ open if for all $\vec{x}_0 \in U$ there is a r (which may depend on $\vec{x}_0$) st $D_r(\vec{x}_0) \subset U$



  ↳ Note: empty set $\varnothing$ considered open, $\mathbb{R}^n$ open

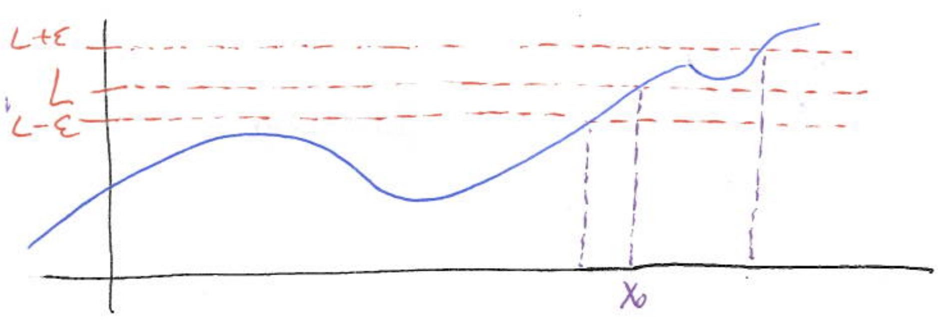
  ↳ use dotted lines to denote open

- **NEIGHBORHOOD**: Mean any open set containing $\vec{x}_0$
- **BOUNDARY POINTS**: Given $A \subset \mathbb{R}^n$, say $\vec{x}$ is a boundary point of A if every neighborhood of $\vec{x}$ contains at least one point in A and at least one point not in A.
- **CLOSED**: A set is closed if it contains all its ~~limit~~ boundary points.

- **LIMIT**: Say the limit of f as $\vec{x}$ approaches $\vec{x}_0$ is L if $f(\vec{x})$ gets closer and closer to L as $\vec{x}$ gets closer to $\vec{x}_0$. Denote $\lim_{x \to x_0} f(x) = L$.

  ↳ Can define in terms of neighborhoods

  ↳ Can do ϵ-δ: $\forall \varepsilon > 0 \; \exists \delta$ st $|\vec{x} - \vec{x}_0| < \delta$ (and $\vec{x} \neq \vec{x}_0$) implies $|f(\vec{x}) - L| < \varepsilon$



EXAMPLE: PROVE $f(x)$ is continuous at $x=3$ if $f(x)=x^2$

"Guess" $L=9$. Given ε find δ st $|x-3|<\delta \Rightarrow |f(x)-9|<L$

Well, $|f(x)-9| = |x^2-9| = |x-3|\cdot|x+3| < \varepsilon$

↳ wlog, assume $\delta<1$ so $|x+3|$ is between 2 and 4

Then $|x-3|\cdot 2 < \varepsilon$ or $|x-3| < \frac{\varepsilon}{2}$

So if we take $\delta < \frac{\varepsilon}{2}$ then $|x-3|<\delta \rightarrow |f(x)-9| < \varepsilon$

- Usually won't argue so rigorously, but good to know "how"

PROPERTIES OF LIMITS

- Uniqueness: $\lim_{x\to x_0} f(x)$ equals b_1 and b_2 then $b_1=b_2$
- Constant: $\lim_{x\to x_0} c\,f(x) = c\lim_{x\to x_0} f(x)$
- Sum/Diff: $\lim_{x\to x_0}\left(f(x)+g(x)\right) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$ if at least one exists on RHS
- Product/Quotient: $\lim_{x\to x_0}\frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$ and $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} f(x)\lim_{x\to x_0} g(x)$

  so long as at least one of RHS exists, and for quotient $\lim_{x\to x_0} g(x)$ is non-zero
- Components $f(\vec{x}) = (f_1(\vec{x}),\dots,f_n(\vec{x}))$. Then $\lim_{x\to x_0} f(\vec{x}) = \vec{b}$ if and only if $\lim_{x\to x_0} f_i(\vec{x}) = b_i$ for $i\in\{1,2,\dots,n\}$

DANGERS: $\lim_{x\to\infty}(x^2-x)$, $\lim_{x\to\infty}(x^2-x^2)$, $\lim_{x\to\infty}(x^2-x^3)$

Do not define $\infty-\infty$, $\pm\infty\cdot 0$; do define $\infty\cdot\infty$, $\infty+\infty$

Section 2.2 (cont)

Continuous Fn: $f: \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\vec{x}_0$ if and only if $\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$. If continuous at each point of defn, say f is continuous (on its domain), else f is discontinuous.

Thm: Properties of Cont Fns: f, g cont at $\vec{x}_0$, c constant

- Constant: $c\,f(\vec{x})$ is cont at $\vec{x}_0$
- Sum/Diff: $f(\vec{x}) \pm g(\vec{x})$ is cont at $\vec{x}_0$
- Prod/Quotient: $f(\vec{x})/g(\vec{x})$ is cont at $\vec{x}_0$ (for quotient need $g(x_0) \neq 0$)
- Component: $f(\vec{x}) = (f_1(\vec{x}), \ldots, f_m(\vec{x}))$ is cont at $\vec{x}_0$ if and only if each f_i is cont at $\vec{x}_0$.

↳ Note limit rules yield continuity rules

↳ To show polynomials are continuous:

↳ One variable:

↳ Start 1, x continuous

↳ get x^n is continuous for any integer $n \geq 0$

↳ get $a_n x^n$ is continuous for any real a_n

↳ get $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ is continuous

↳ Generalizes to several variables

↳ Important that is a finite sum

Section 2.2 (Cont)

New feature emerges in higher dimensions: limit must be independent of path. In one-dimension, essentially only ~~one~~ two possible paths: $\longrightarrow$ or $\longleftarrow$. MANY more in $\mathbb{R}^n$.

EXAMPLE: $f(x,y) = \dfrac{x^2}{x^2+y^2}$. Does the limit exist as $(x,y) \to (0,0)$?

Take path $x=0$: $\lim\limits_{y\to 0} f(0,y) = \lim\limits_{y\to 0} 0 = 0$

Take path $y=0$: $\lim\limits_{x\to 0} f(x,0) = \lim\limits_{x\to 0} \dfrac{x^2}{x^2} = \lim\limits_{x\to 0} 1 = 1$

↳ as unequal, limit does not exist

↳ Note: just because get the same limit along two paths does not mean limit exists. Can disprove by example, not prove by example (Consider 16/64...)

EXAMPLES: $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2}$, $\lim\limits_{x\to\infty} \dfrac{x^3-x^2+17x}{x^4-x^2+2x}$, $\sin\frac{1}{x}$ and $x\sin\frac{1}{x}$

Review L'Hopital, ask $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$

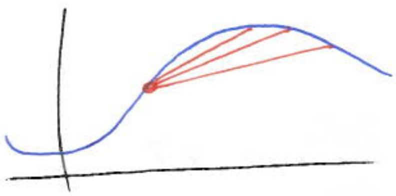
Homework: #4, #8ab, #17, #21

Suggested: #8c, #9, #10, #25, #26, #27

SECTION 2.3: DIFFERENTIATION

Defn of the deriv (one variable)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \quad \text{or} \quad f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$



Interpretation: average speed

PARTIAL DERIV

$$\frac{\partial f}{\partial x_j}(x_1,\dots,x_n) = \lim_{h\to 0} \frac{f(x_1,x_2,\dots,x_j+h,\dots,x_n) - f(x_1,\dots,x_n)}{h}$$

$$= \lim_{h\to 0} \frac{f(\vec{x}+h\vec{e}_j) - f(\vec{x})}{h}$$

↳ Just treat all other variables as constants

Example: Find partials of $f(x,y) = x\cos(xy)$

OUTSTANDING EXAMPLE: $f(x,y) = (xy)^{1/3}$

↳ $\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0 = \frac{\partial f}{\partial y}(0,0)$

Let $g: \mathbb{R}\to\mathbb{R}^2$ be given by $g(x) = (x,x)$

Let $A(x) = f(g(x)) = (f\circ g)(x) = x^{2/3}$

↳ f and g differentiable

$$A'(x) = \lim_{h\to 0} \frac{A(0+h)-A(0)}{h} = \lim_{h\to 0} \frac{h^{2/3}}{h} = \text{undefined!}$$

↳ COMPOSITION OF DIFF IS NOT NECC DIFF

↳ CHAIN RULE WILL BE EVEN HARDER

↳ PROBLEM HERE: NO GOOD TANGENT PLANE AT (0,0)

Section 2.3 (cont)

Linear Approximations

↳ Very important: locally complex fns well approx with simple fns

↳ tangent line: $y = f(a) + f'(a)(x-a)$

↳ Size of error?

↳ MVT: $f(x) = f(a) + f'(c)(x-a)$

Thus $|f(x) - y| = |f'(c) - f'(a)| \cdot |x-a|$

$$\le \left| 2 \max_{w \in [a,c]} f'(w) \right| \cdot |x-a|$$

↳ better estimate if f'' exists

↳ Generalize to higher dimensions

Tangent Plane:

$$z = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0)$$

Deriv in one-var: $\displaystyle \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0$

Defn of the deriv: Open $U \subset \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}^m$ is diff at $\vec{x}_0$ if partial derivs exist at $\vec{x}_0$, and if $T = Df(\vec{x}_0)$ is the $m \times n$ matrix with elements $\partial f_i / \partial x_j (\vec{x}_0)$ satisfies

$$\lim_{\vec{x} \to \vec{x}_0} \frac{\| f(\vec{x}) - f(\vec{x}_0) - T(\vec{x} - \vec{x}_0) \|}{\| \vec{x} - \vec{x}_0 \|} = 0,$$

with $\vec{x} - \vec{x}_0$ a column vector. Call T the deriv of f at x_0 or the matrix of partial derivs of f at $\vec{x}_0$

- Example in 2-dim: $f: \mathbb{R}^2 \to \mathbb{R}$ is diff at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0) - \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y-y_0)}{\|(x,y)-(x_0,y_0)\|}$$

tends to 0. In other words, locally tangent plane approxs well.

- Example: $f(x,y) = x^2 + y^4 + e^{xy}$ at $(1,0,2)$

  Write it as $z = f(x,y)$, with $z_0 = f(x_0,y_0) = f(1,0) = 2$

  $\frac{\partial f}{\partial x} = 2x + ye^{xy} \Rightarrow \frac{\partial f}{\partial x}(1,0) = 2$

  $\frac{\partial f}{\partial y} = 4y^3 + xe^{xy} \Rightarrow \frac{\partial f}{\partial y}(1,0) = 1$

  Tangent plane is $z = 2 + 2(x-1) + 1\cdot(y-0)$

SPECIAL CASE: GRADIENT

$f: \mathbb{R}^n \to \mathbb{R}$ Then $Df(\vec{x}) = \left[\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n}\right]$ is a $1\times n$ matrix, form corresponding vector $\nabla f = \text{grad}(f) = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$, called the gradient of f.

Example: $f(x,y,z) = x^2+y^2+z^2$ then $\nabla f = (2x, 2y, 2z)$

Note $Df(\vec{x})(\vec{h}) = \nabla f(\vec{x}) \cdot \vec{h}$

APPLICATION: ESTIMATING FUNCTIONS

$f(x,y) = xe^{xy}$ estimate $f(.98, -.01)$

↳ take $(x_0,y_0) = (1,0)$, $(x,y) = (.98,-.01)$, $\frac{\partial f}{\partial x} = e^{xy} + xye^{xy} \xrightarrow[y=0]{x=1} 1$, $\frac{\partial f}{\partial y} = x^2 e^{xy} \to 1$

So $f(.98,-.01) \approx 1 + 1(.98-1) + 1(-.01-0) = .97$ Answer is $.97043$

SECTION 2.3 (CONT)

THM: $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ diff at $\vec{x}_0 \implies f$ is cont at $\vec{x}_0$

Proof: $f(\vec{x}) = f(\vec{x}_0) + Df(\vec{x}_0)(\vec{x}-\vec{x}_0) + E_{\vec{x}_0}(\vec{x})$

with $\lim_{x \to \vec{x}_0} \|E_{\vec{x}_0}(\vec{x})\| / \|\vec{x}-\vec{x}_0\| = 0$

THM: $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ st all partial derivs $\partial f_i / \partial x_j$ exist and continuous in a nbhood of $\vec{x} \in U$. Then f is differentiable at $\vec{x}$.

↳ Recall $f(x,y) = (xy)^{1/3}$

↳ partial derivs not continuous at $(0,0)$: $\frac{\partial f}{\partial x} = \frac{1}{3} x^{-\frac{2}{3}} y^{\frac{1}{3}}$

and if $y = x$ then $\frac{\partial f}{\partial x}(x,x) = \frac{1}{3} x^{-1/3}$

Following valid: Continuous Partial Derivatives $\implies$ Function is Differentiable $\implies$ Partial Derivatives Exist

Each converse statement, obtained by reversing arrows, can fail!

Say a function is C^1 if it is continuously differentiable, C^2 if twice continuously differentiable, and so on.

NOTE: $f(x,y) = (xy)^{1/3}$ is not differentiable at $(0,0)$

Tangent plane is $z=0$, but $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{f(x,y)}{\|(x,y)\|}$ undefined

SECTION 2.3 (CONT)

PROOF PARTIALS EXIST AND CONT $\Longrightarrow$ FN IS DIFFERENTIABLE

- Will do $n=2$, generalization possible
- Highlights an important technique: adding zero.
- Natural guess for derivative: try it:

$$f(x,y) - f(0,0) - \left[\frac{\partial f}{\partial x}(0,0)\right]x - \left[\frac{\partial f}{\partial y}(0,0)\right]y$$

$$= \left\{\underbrace{f(x,y) - f(0,y)}_{MVT} - \left[\frac{\partial f}{\partial x}(0,0)\right]x\right\} + \left\{\underbrace{f(0,y) - f(0,0)}_{MVT} - \left[\frac{\partial f}{\partial y}(0,0)\right]y\right\}$$

$$= \left\{\left[\frac{\partial f}{\partial x}(c,y)\right]x - \left[\frac{\partial f}{\partial x}(0,0)\right]x\right\} + \left\{\left[\frac{\partial f}{\partial y}(0,\tilde{c})\right]y - \left[\frac{\partial f}{\partial y}(0,0)\right]y\right\}$$

$$= \underbrace{\left[\frac{\partial f}{\partial x}(c,y) - \frac{\partial f}{\partial x}(0,0)\right]}_{\to 0 \text{ (by continuity)}}x + \underbrace{\left[\frac{\partial f}{\partial y}(0,\tilde{c}) - \frac{\partial f}{\partial y}(0,0)\right]}_{\to 0 \text{ (by continuity)}}y$$

As $\dfrac{\|x\|}{\|(x,y)\|}$ and $\dfrac{\|y\|}{\|(x,y)\|}$ bounded, above tends to zero even after dividing by $\|(x,y)\| = \|(x,y)-(0,0)\|$. $\blacksquare$

Homework: #2ab, #4ab, #5, #7c, #12a (linear approx), #13a

Suggested: #3, #4cde, #10, #15, #18, #19

SECTION 2.4: INTRO TO PATHS AND CURVES

Path in $\mathbb{R}^n$ is a map $c : [a,b] \to \mathbb{R}^n$. Collection C of points $c(t)$ is a curve with endpoints $c(a)$, $c(b)$.
The path c parametrizes the curve C.

↳ Ex: $n=2$: $c(t) = (x(t), y(t))$

Ex: line: $c(t) = \vec{a} + t\vec{v}$

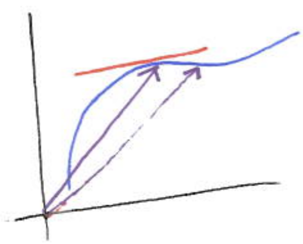
Ex: circle: $c_1(t) = (\cos t, \sin t)$ $\quad t \in [0, 2\pi]$

$c_2(t) = (\cos 2t, \sin 2t)$ $\quad t \in [0, \pi]$

$c_3(t) = (\sin t, \cos t)$ $\quad t \in [0, 2\pi]$

↳ different paths parametrize same curve

Velocity vector: If c is a differentiable path then velocity of c at time t is $c'(t) = \lim_{h \to 0} \frac{c(t+h) - c(t)}{h}$



↳ denote speed by $s(t) = \|c'(t)\|$

↳ tangent line at time t_0 is $l(t) = c(t_0) + c'(t_0)(t - t_0)$

Homework: #1, #7, #15

Suggested: #3, #6, #13, #17

SUPPLEMENT TO SECTION 2.4

Below are some items related to the lecture on Section 2.4. This is just a quick sketch of the additional items from the lecture not in the quick lecture note summary.

Applications:
- Planetary motion (orbits, probes)
- Ballistics (projectiles, cannon balls, baseball)

Example: Parametrizing Circle

- Regard as level set of $f(x,y)$ with value r^2 (here $f(x,y) = x^2 + y^2$).
- Want to solve $x(t)^2 + y(t)^2 = r^2$ for circle
- What do we need to parametrize a circle?
  - ↳ radius r
  - ↳ Center (x_0, y_0)
  - ↳ speed and direction: velocity
  - ↳ starting point

Ex: Circle of radius 2, center $(0,0)$, start at $(2,0)$, counterclockwise

↳ Soln: $\vec{c}(t) = (x(t), y(t)) = (2\cos t, 2\sin t)$

↳ check: clear $x(t)^2 + y(t)^2 = 4$

must show get all points as time passes

↳ show given any point there is a t

SUPPLEMENT TO SECTION 2.4 (CONT)

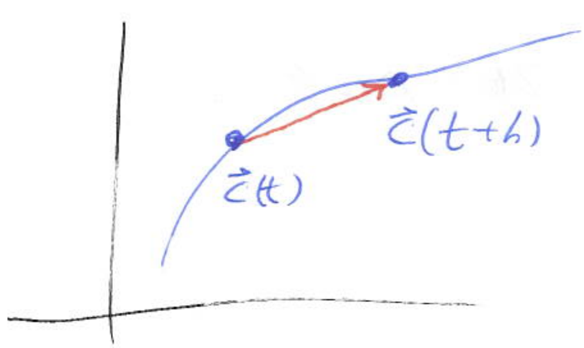
Consider: • $\vec{c}(t) = (2\cos t,\ 2\sin t)$

$$\vec{d}(t) = (2\cos(-t),\ 2\sin(-t))$$
$$= (2\cos t,\ -2\sin t)$$
$$\vec{e}(t) = (2\cos(2\pi t),\ 2\sin(2\pi t))$$

↳ Physically all trace out same circle
  ↳ $\vec{c}$ and $\vec{e}$ do it clockwise, $\vec{d}$ counterclockwise

↳ Note $\vec{e}$ is faster than $\vec{c}$ and $\vec{d}$: takes 1 unit of time to trace out, as opposed to 2π for $\vec{c}, \vec{d}$
  ↳ leads to studying speed of a parametrization
    ↳ often useful travel unit speed
      ↳ lots of eqs involve speed, if unit and forget not a big deal
      ↳ ex: angle b/w $\vec{v}$ and $\vec{w}$ is: $\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos\theta$
        bad if forget lengths unless equal 1.

Tangent Vector



$\vec{c}(t+h) - \vec{c}(t)$ is approx distance traveled in time h, gets better as time shrinks to 0. Thus instantaneous speed is well approx by $\|\vec{c}(t+h) - \vec{c}(t)\| / h$, and speed $= \|\vec{c}\,'(t)\|$

Supplement to Section 2.4 (cont)

Want to reparametrize to unit speed

Consider $\vec{c}(t) = (2\cos t, 2\sin t)$

↳ have $\vec{c}\,'(t) = (-2\sin t, 2\cos t)$

$$\|\vec{c}\,'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t}$$
$$= \sqrt{4} = 2$$

Thus traveling at say 2 units/sec. How can we adjust to travel at 1 unit/sec?

<u>Attempt 1</u>: $\vec{h}_1(t) = \frac{1}{2}\vec{c}(t) = (\cos t, \sin t)$

↳ natural to try. Do get $\|\vec{h}_1\,'(t)\| = 1$, but no longer a circle of radius 2! Cannot change output: must trace same physical shape.

<u>Attempt 2</u>: $\vec{h}_2(t) = \vec{c}(at) = (2\cos(at), 2\sin(at))$

↳ natural: adjusting how rapidly time passes!

$\vec{h}_2\,'(t) = (-2a\sin(at), 2a\cos(at))$

$\|\vec{h}_2\,'(t)\| = \sqrt{4a^2\sin^2(at) + 4a^2\cos^2(at)} = 2|a|$

↳ Thus $a = 1/2$ (orientation preserving) or $a = -1/2$ (reversing)

SUPPLEMENT TO SECTION 2.4 (CONT)

Example: parametrize level set of value 4 to $f(x,y)=x^2+4y^2$

- If had $x^2+y^2=4$ a circle, should be similar

Try $x(t)=a\cos t$ $y(t)=b\sin t$

Get $\vec{c}(t)=(a\cos t, b\sin t)$

↳ $x(t)^2+4y(t)^2=a^2\cos^2 t+4b^2\sin^2 t$

↳ if $a^2=4b^2$ then have same number of cosines and sines ∴ Pythagoras simplifies

If $a^2=4b^2=4$ then works!

Thus soln is $a=\pm 2$, $b=\pm 1$

↳ Ex: take $\vec{c}(t)=(2\cos t, \sin t)$

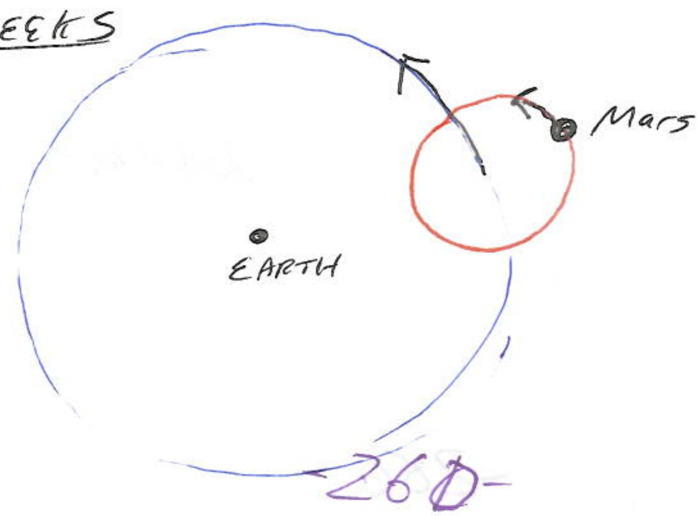
↳ parametrizes ellipse!

↳ Question: Constant speed?

$\vec{c}\,'(t)=(-2\sin t, \cos t)$

$\|\vec{c}\,'(t)\| = \sqrt{4\sin^2 t+\cos^2 t} = \sqrt{1+3\sin^2 t}$: NO!

BEAT THE GREEKS



circles on circles....

Notes: two days on chain rule

First day proved diff results / stated results

↳ simple fns, compound then

↳ diff laws: const, sum/diff, prod/quotient

↳ proved ~~quot~~ prod: highlight add 0

Examples of composition

↳ art forgery & composition

Sabermetrics runs created → Oy Ding

Chain Rule

↳ Proof in 1 dim: emph mult by 1, get denom right, see deriv

Do 1st special case $h(t) = f(c(t)) \rightarrow \frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \dots$

2nd special case: $h(t) = f(g(u,v)) \rightarrow \frac{\partial h}{\partial u} =$ ______

State general case, motivate multiplying matrices by dot product

do special case $f(u,v) = \frac{v^2}{u^2+v^2}$ $u = e^{-x-y}$, $v = e^{xy}$

↳ see chain can be good

↳ also do example special case:

$c(t) = (\cos t, \sin t)$

$f(x,y) = 3x^2 + y$

↳ tell story: temp as flow

2010: Days 10 and 11