CHAPTER 4: VECTOR VALUED FUNCTIONS

We will skip this chapter (ie, we will not cover this material in the main lectures)

Supplemental notes from lectures I gave on these topics in a 15 week class follow.

Chapter 4: Vector Valued Fns

4.1. Acceleration + Newton's Second Law

$$\vec{c}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad c'(t) = \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}$$

Rules: $b(t), c(t)$ paths, $p(t), q(t)$ scalars

↳ Sum, scalar mult: $p'c + pc'$

dot product: $b' \cdot c + b \cdot c'$

cross product $b' \times c + b \times c'$ ($\mathbb{R}^3$!)

chain $c'(q(t)) \cdot q'(t)$

Ex: Dot product: $f(t) = b(t) \cdot c(t) = \sum_{i=1}^{n} b_i(t) c_i(t)$

$$f'(t) = \sum_{i=1}^{n} \left[b_i'(t) c_i(t) + b_i(t) c_i'(t) \right] = \cdots$$

Lemma: $\|c(t)\|$ constant $\to c'(t) \perp c(t)$

Proof: Easier study $f(t) = \|c(t)\|^2 = c(t) \cdot c(t)$

Then $0 = f'(t) = 2c(t) \cdot c'(t)$

Defn: diff path regular at t_0 if $c'(t_0) \neq 0$

Defn: Velocity $v(t) = c'(t)$, acceleration $a(t) = v'(t) = c''(t)$

Newton: $F = ma$ (book: $P = mf$!)

HW: pg 273: #1, #7, #13

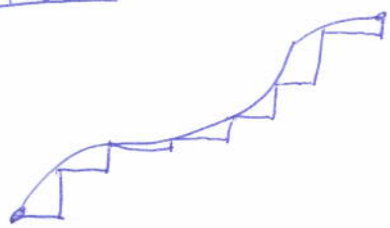
Suggested: #14, #15, #20, #21

Extra Credit: prove in Newton's Law of Gravity may assume all mass at center if uniform. Hint .
enough to do for infinitesimal planar ring.

4.2. Arc Length

(False) Thm: Two curves w/ same endpoints have same length!

(False) Proof



in limit as take more pieces, better approx 

move all — down, all | right

length $= \Delta x + \Delta y$!

Absurd:  $\uparrow$ # wiggles, $\uparrow$ approx length

What is good approx to length is hypotenuse!

need some higher analysis to rigorize hypotenuse lengths by Pythagoras

$$\sum_{k=1}^{m} \sqrt{(x_1(t_k) - x_1(t_{k-1}))^2 + \cdots + (x_n(t_k) - x_n(t_{k-1}))^2} \quad \downarrow \text{MVT}$$

$$= \sum_{k=1}^{m} \sqrt{\sum_{i=1}^{n} x_i'(c_{k,i})^2 (t_k - t_{k-1})^2}$$

$$= \sum_{k=1}^{m} \sqrt{\sum_{i=1}^{n} x_i'(c_{k,i})^2}\ (t_k - t_{k-1})$$

$$\downarrow \lim m \to \infty$$

$$= \int_{t=a}^{b} \sqrt{\sum_{i=1}^{n} x_i'(t)}\ dt = \int_{t=a}^{b} \|c'(t)\|\, dt := s(t)$$

Also write $\int_a^b \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \cdots + \left(\frac{dx_n}{dt}\right)^2}\ dt$

idea: using ave speed on sub-intervals. If next deriv exists, by MVT small error (relative to size!)

uniform cont

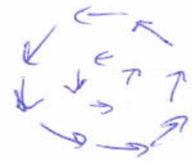
HW: pg 281: #1, #11, #13

Suggested: #3, #12, #14, #15, #16, #17, #19

4.3. Vector Fields

Defn: Vector field: $F: A \subset \mathbb{R}^n \to \mathbb{R}^n$ assigns to each $\vec{x} \in A$ a vector $F(\vec{x})$

Scalar field: $F: A \subset \mathbb{R}^n \to \mathbb{R}$

Ex: $V(x,y) = \left(\frac{y}{\cancel{x^2+y^2}}, -\frac{x}{\cancel{x^2+y^2}}\right)$  note $V(x,y) \perp \vec{r}$

Gradient Vector Field: $F(\vec{x}) = \nabla f(\vec{x}) = \sum \frac{\partial f}{\partial x_i} \hat{e}_i$

$$\boxed{\text{Del Operator: } \nabla = \sum \hat{e}_i \, \partial/\partial x_i}$$

↳ ex: $F = -G\frac{mM}{r^3}\vec{r} = -\nabla V(\vec{r})$, $V(\vec{r}) = -G\frac{mM}{r}$

↳ call V the potential

↳ also useful electromagnetism

Not all vector fields have a potential

↳ $V(x,y) = (y, -x)$

Suppose $V(x,y) = \nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (y, -x)$

As C^1 fns, f has cont 1st + 2nd derivs

But $\partial^2 f/\partial x\partial y = -1$ and $\partial^2 f/\partial y \partial x = 1$ contra

HW: pg 293: #5

Suggested: #2, #7, #20

4.4. DIVERGENCE + CURL

Defn: del operator: $\nabla = \sum_{i=1}^{n} \hat{e}_i \frac{\partial}{\partial x_i}$

↳ note $grad(f) = \nabla f$

Defn: Divergence: $div\, F = \nabla \cdot F \qquad F = (F_1, \dots, F_n)$

↳ will discuss in great detail in Chap 8, see historical note pg 308

Measures net flow across surfaces

↳ ex  take y-z plane, normal $\hat{e}_x = \hat{i}$

only change in x-dir matters

Defn: $curl\, F = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$ this is 3-dim!

↳ Chapt 8: measures rotation

Ex: $F(x,y) = (-y, x) \perp \vec{r}$ (better: $(-y, x, 0) \perp \vec{r}$)

$\nabla \cdot F = (0,0)$: no divergence

$\nabla \times F = (0,0,2)$: rotation

↳ like $\cup$ and $\cap$ to remember min/max in 1-var

Thm: $\nabla \times (\nabla f) = 0$: direct calc @ mixed derivs equal
(if f is C^2)

↳ Quick test if gradient field: $\nabla \times F \neq \vec{0} \Rightarrow$ no (other dir Chapt 8)

Thm: $div\, curl\, F = \nabla \cdot (\nabla \times F) = 0$ if F is C^2

Laplace: $\nabla \cdot \nabla f$, ~~lots~~ $= \sum \partial^2 f / \partial x_i^2$

Lots identities pg 306, Historical Note + Gibbs

HW: pg 310: #3, #15, #31

Suggested: #25

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