CHAPTER 5: DOUBLE AND TRIPLE INTEGRALS

Generalize integrating on a line

Easiest is integrating over a box, or cube, ...

In many cases, double and triple integrals become iterated integrals

We'll review the proof of the Fund Thm of Calc (Riemann Sums) in one variable and then briefly discuss generalizations. Won't prove for widest class of functions to avoid appealing to Real Analysis. Key ingredient in proof is the Mean Value Theorem:

If g is cont and differentiable on $[a,b]$ then there is a c st $\frac{g(b)-g(a)}{b-a} = g'(c)$ with $c \in [a,b]$.

The next three pages are notes of mine on proving FTC.

ELEMENTARY AREA COMPUTATIONS

Know area of rectangle, right triangle , general triangle 

↳ area under curve: approx with rectangles

inscribe + circumscribe: squeeze intuitive value between

<u>Ex: $f(x) = x$, $0 \le x \le 1$</u>

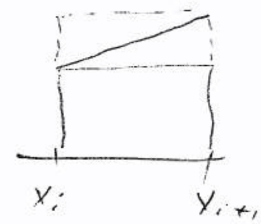
↳ clear area just $1/2$ 

Divide $[0,1]$ into n equal pieces, $0 = x_0 \le x_1 \le \cdots \le x_n = 1$

so $x_i = i/n$ and $x_{i+1} - x_i = \frac{1}{n}$

Let $I_i = [x_i, x_{i+1}]$

on I_i, $f(x)$ satisfies $f(x_i) \le f(x) \le f(x_{i+1})$



so $\sum_{i=0}^{n-1} f(x_i) \cdot \frac{1}{n} \le \text{Area} \le \sum_{i=0}^{n-1} f(x_{i+1}) \frac{1}{n}$

$\sum_{i=0}^{n-1} \frac{i}{n}\frac{1}{n} \le \text{Area} \le \sum_{i=0}^{n-1} \frac{i+1}{n}\frac{1}{n} = \sum_{i=0}^{n-1} \frac{i}{n^2} + \frac{n}{n^2}$

<u>Claim</u>: $\sum_{i=0}^{m} i = \frac{m(m+1)}{2}$ (induction, match largest + smallest, ...)

Thus $\frac{1}{n^2}\frac{(n-1)n}{2} \le \text{Area} \le \frac{1}{n^2}\frac{(n-1)n}{2} + \frac{n}{n^2}$

take $\lim_{n\to\infty}$, get $\frac{1}{2} \le \text{Area} \le \frac{1}{2}$ so Area is $1/2$

<u>Ex: $f(x) = x^2$, $0 \le x \le 1$</u>

↳ Before $f(x_i) = i/n$, now $f(x_i) = (i/n)^2$

Get $\sum_{i=0}^{n-1} \frac{i^2}{n^2}\frac{1}{n} \le \text{Area} \le \sum_{i=0}^{n-1} \frac{i^2}{n^2}\frac{1}{n} + \frac{n^2}{n^3}$

<u>Claim</u>: $\sum_{i=0}^{m} i^2 = \frac{m(m+1)(2m+1)}{6}$

↳ yields $\frac{1}{3} \le \text{Area} \le \frac{1}{3}$ after taking limits   HW: 3, 9, 19, 22, 39

<u>Generalizations</u>   Suggested: 52, 53

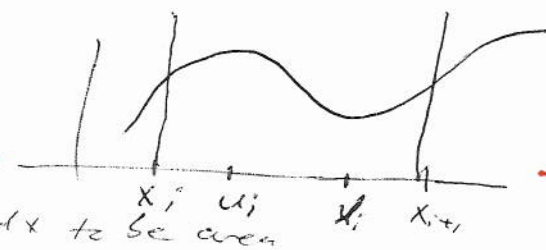
↳ $f(x) = x^3$: need $\sum_{i=0}^{m} i^3 = \frac{m^2(m+1)^2}{4}$

↳ have to prove "hard" sums each time

↳ what of more general f, say $f(x) = x^2 \sin x$?

↳ $[a,b]$ set $x_i = a + \frac{b-a}{n}$

Will not prove in greatest generality
↳ each n partition $[a,b]$ into n pieces, find upper & lower sums, limit
  ↳ will assume all pieces equal
  ↳ will assume $[a,b]=[0,1]$



FTC: Let f be cont on $[a,b]$. Set $\int_a^b f(x)\,dx$ to be area under f from a to b. If F is any anti-deriv of f then $\int_a^b f(x)\,dx = F(b)-F(a)$ (Fund Thm of Calculus)

↳ Note: G any other anti-deriv: $F(x)=G(x)+C$, so $F(b)-F(a)=G(b)-G(a)$
↳ will prove assuming f' ~~cont~~ bounded: simplifies proof Talk about bounded

Proof (The One With The MVT)
↳ $I_i := [x_i, x_{i+1}]$: let max be at u_i, min at l_i
  so $\forall x \in I_i$, $f(l_i) \le f(x) \le f(u_i)$
  so $L(n) = \sum_{i=0}^{n-1} f(l_i)\frac{1}{n} \le \text{Area} \le \sum_{i=0}^{n-1} f(u_i)\frac{1}{n} = U(n)$

Claim: $\lim_{n\to\infty} U(n)-L(n)=0$
  Proof: Equals $U(n)-L(n) = \frac{1}{n}\sum_{i=0}^{n-1}[f(u_i)-f(l_i)]$
  By MVT, $= \frac{1}{n}\sum_{i=0}^{n-1} |f'(c_i)||u_i-l_i|$
  $\le \frac{1}{n}\sum_{i=0} B\cdot\frac{1}{n} = \frac{B}{n} \longrightarrow 0$

Let $x_i^* \in I_i$ be any seq of points
↳ $L(n) \le \sum_{i=0}^{n-1} f(x_i^*)\frac{1}{n} \le U(n)$; choose clever seq
  Apply MVT to F on I_i: choose x_i^* st $F(x_{i+1})-F(x_i)=F'(x_i^*)(x_{i+1}-x_i)$, $F'(x_i^*) = f(x_i^*)$
  so $L(n) \le \sum_{i=0}^{n-1} f(x_i^*)\frac{1}{n} = \sum_{i=0}^{n-1} F(x_{i+1})-F(x_i) \le U(n)$
  └ telescoping: is $F(1)-F(0)$ or $F(b)-F(a)$

so $L(n) \le F(1)-F(0) \le U(n)$
↓ Area   Thus goes to area   ↓ Area

Call f <u>Riemann Integrable</u> if above method works

Not all f are RI: $f(x) = 0$ if $x \in \mathbb{Q}$, 1 if $x \notin \mathbb{Q}$

↳ <u>Lebesgue Integral</u>: agrees when RI exists, generalizes

↳ Throw down coins: RI: sum in order, LI: group pennies, nickels...

<u>Average Values</u>

If f integrable on $[a,b]$, ave value is $\frac{1}{b-a} \int_a^b f(x)\,dx$

↳ if f cont, $\exists\ \bar{x} \in [a,b]$ st $f(\bar{x}) =$ Ave Value

HW: Sec 5.6: 1, 13, 27, 45

Suggested: Sec 5.6: 41, 43, 66

*<u>FTC</u>: Define $F(x) = \int_a^x f(x)\,dx$ Then $F'(x) = f(x)$

<u>Extra Credit</u>: Say $G(x) = \int_a^{x^3} f(x)\,dx$. Express $G'(x)$ in terms of nice fns related to this problem.

<u>5.5 Evaluation of Integrals</u>

① Constant: $\int_a^b c\,dx = c(b-a)$

② Multiple: $\int_a^b c f(x)\,dx = c \int_a^b f(x)\,dx$

③ Sum/Diff: $\int_a^b (f(x) \pm g(x))\,dx =$

④ Interval Union: $a < c < b$: $\int_a^b f = \int_a^c f + \int_c^b f$

⑤ Comparison: $f \le g$ on $[a,b]$ Then $\int_a^b f \le \int_a^b g$

↳ special case: $m \le f(x) \le M \Rightarrow m(b-a) \le \int_a^b f \le M(b-a)$

<u>Do Many Examples!</u>

HW: 1, 13, 33, 54

Suggested: 59, 60

SECTION 5.1: CAVALIERI'S PRINCIPLE

Way to do volumes in special cases

SLICE METHOD / CAVALIERI'S PRINCIPLE

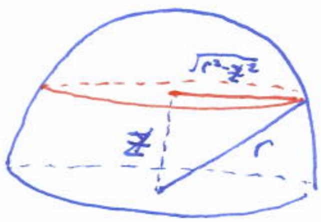
S a solid, and for x st $a \le x \le b$ we have a family of parallel planes P_x such that

(1) S lies between P_a and P_b

(2) Area of slice of S cut by P_x is $A(x)$

Then $vol(S) = \int_a^b A(x)\,dx$.

Ex:



Hemisphere of radius r

Have z range from 0 to r

At height z, radius is $\sqrt{r^2-z^2}$

Thus $A(z) = \pi(\text{radius at } z)^2 = \pi(r^2-z^2)$

$$\text{So Vol(Hemisphere of radius } r) = \int_0^r \pi(r^2-z^2)\,dz$$

$$= \pi r^2 z\Big|_0^r - \pi \frac{z^3}{3}\Big|_0^r$$

$$= \frac{2}{3}\pi r^3$$

Thus vol of sphere of radius r is $\frac{4}{3}\pi r^3$! Fast!

Idea is doing iterated integrals: first found area in the planes, then integrated that.

SECTION 5.1 (CONT)

Can do volumes as double integral

Let $f(x,y)$ be height at x,y

Volume is $\iint_R f(x,y)\,dx\,dy$

If use planes perpendicular to x-axis:

$$\iint_R f(x,y)\,dA = \int_{x=a}^{b} \left[\int_{y=c}^{d} f(x,y)\,dy \right] dx$$

If use planes perpendicular to y-axis:

$$\iint_R f(x,y)\,dA = \int_{y=c}^{d} \left[\int_{x=a}^{b} f(x,y)\,dx \right] dy$$

For nice regions, these must be equal.

What if not a volume: can we still change order?

Ex: $\int_0^1 \int_0^1 x e^{xy}\,dy\,dx$ $\qquad \int_0^1\int_0^1 x e^{xd}\,dx\,dy$

↳ arg! integrate by parts

$$= \int_0^1 \left[\int_0^1 e^{xy} x\,dy \right] dx$$

$$= \int_0^1 e^{xy} \Big|_0^1 dx$$

$$= \int_0^1 (e^{y} - 1)\,dx = e^{y} - 2$$

$\boxed{\text{Changing orders can lead to a simpler computation!}}$

Homework: #1ac, #6, #7

Suggested: #1bd, #~~2~~ #3, #9

SECTION 5.2: DOUBLE INTEGRAL OVER A RECTANGLE

Wont go into proof in great detail

Natural generalization of Rienam Sums

Proofs easier if assume derivative of function is **bounded**; We say f is bounded by B on the rectangle R if for all $(x,y) \in R$ we have $|f(x,y)| \leq B$.

Often only need to know a B exists, not optimal.

Ex: $f(x) = 3x^2 - x^4 + e^x \cos x$ for $-2 \leq x \leq 1$

$$|f(x)| \leq 3|x|^2 + |x|^4 + e^{|x|}|\cos x|$$

$$\leq 3 \cdot 4 + 16 + e^2 \cdot 1 \leq 50$$

very wasteful

↳ actual bound is under 5!

5.2. The Double Integral over a Rectangle

- Standard props
  - ↳ linearity
  - ↳ homogeneity (const mult)
  - ↳ monotonic: $f \le g$
  - ↳ additivity

- Interchanging orders!

$$\sum_m \sum_n a_{mn} \neq \sum_n \sum_m a_{mn}$$

$$\begin{array}{cccc} \uparrow & & & \\ 0 & 0 & -1 & \\ 0 & -1 & +1 & \\ -1 & +1 & 0 & \\ +1 & 0 & 0 & \rightarrow \end{array}$$

↳ (1) infinite range

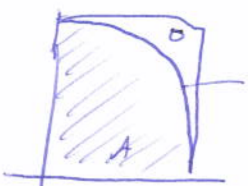
(2) absolute value not integrable

Thm (Fubini): If f integrable on finite R Then can interchange orders

↳ can generalize to bounded fns + discont

↳ see book for proof

Ex:



$$\iint_A \sqrt{1-x^2}\, dA$$

$$= \int_{x=0}^{1} \left[\int_{y=0}^{\sqrt{1-x^2}} \sqrt{1-x^2}\, dy \right] dx = \int_{y=0}^{1} \left[\int_{x=0}^{\sqrt{1-y^2}} \sqrt{1-x^2}\, dx \right] dy$$

one easy, one hard

$$\left(\frac{\sin(2\arcsin x)}{4} + \frac{\arcsin x}{2} \right)' = \sqrt{1-x^2} \;!$$

What is $\iint_A \sqrt{1-x^2-y^2}\, dA$? Vol sphere, realize is easy!

↳ will get with change of variable later.

H(w): Pg 339: #1ab, #8, #9, #12

Suggested: #1cd, #11

5.2. The Double Integral over a Rectangle

Integrate over rectangles first

Partition $[a,b] \times [c,d]$ into $R_{jk} = [x_j, x_{j+1}] \times [y_k, y_{k+1}]$, $\Delta x = \frac{b-a}{n}$

$\Delta y = \frac{d-c}{n}$

Defn: Definite $\int$: If for any seq $\vec{c}_{jk}^{\,(n)} \in R_{jk}$ have

$$S_n = \sum_j \sum_k f(\vec{c}_{jk}^{\,(n)}) \overbrace{\Delta x \Delta y}^{\Delta A}$$ converges then call

number (limit) $\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy$ and say f integrable over R.

Thm: f is $C^1 \to f$ integrable on rectangles

Proof: $\vec{c}_{jk}^{\,(n)}, \vec{d}_{jk}^{\,(n)} \in R_{jk}$, say (x_{jk}, y_{jk}) and $(\tilde{x}_{jk}, \tilde{y}_{jk})$

Then fn $|f(\tilde{x}_{jk}, \tilde{y}_{jk}) - f(x_{jk}, y_{jk})|$

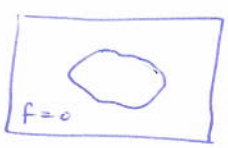
$$\le |f(\tilde{x}_{jk}, \tilde{y}_{jk}) - f(x_{jk}, \tilde{y}_{jk})| + |f(x_{jk}, \tilde{y}_{jk}) - f(x_{jk}, y_{jk})|$$

$$\le \left|\frac{\partial f}{\partial x}(\hat{x}_{jk}, \tilde{y}_{jk})\right| \Delta x + \left|\frac{\partial f}{\partial y}(x_{jk}, \hat{y}_{jk})\right| \Delta y$$

$$\le B\Delta x + B\Delta y \quad (f' \, C^1 \to f' \text{ bounded})$$

so diff b/w $\le \sum_j \sum_k B((\Delta x)^2 \Delta y + \Delta x (\Delta y)^2) \le \text{Const}/n \to 0$

Thm: $f: R \to \mathbb{R}$ bounded, set of points discontinuity finite union of graphs of cont fns. Then f integrable over R



useful situations like this

5.3. The Double Integral over more general regions

<u>Defn: y-simple</u> cont $\varphi_1, \varphi_2 : [a,b] \to \mathbb{R}$ st $\varphi_1(x) \le \varphi_2(x)$

$D = \{(x,y) : x \in [a,b],\ y \in [\varphi_1(x), \varphi_2(x)]\}$: D is y-simple

↳ curves and line segments boundary

↳ x-simple if cont fns $\psi_1, \psi_2 : [c,d] \to \mathbb{R}$

↳ simple: x-simple + y-simple (or <u>elementary region</u>)

↳ boundary is type set of discont from earlier thms

<u>Defn: integral over elementary region</u>

D elemental, rect $R \supset D$, $f^*(x,y) = \begin{cases} f(x,y) & (x,y) \in D \\ 0 & \text{other} \end{cases}$

Then $\iint_D f\,dA = \iint_R f^*\,dA$

↳ if D is y-simple $= \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy \right] dx$

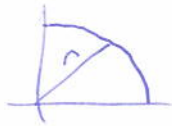
HW: pg 347: #2a,b, #3, #15

Suggested: #9, #12, #16

5.4. CHANGING THE ORDER OF INTEGRATION

Already discussed often easier to do one-way than other

$$\iint_D f\,dA = \int_{x=a}^{b}\int_{y=\varphi_1(x)}^{\varphi_2(x)} f\,dy\,dx = \int_{y=c}^{d}\int_{x=\psi_1(y)}^{\psi_2(y)} f\,dx\,dy$$

Ex: $\iint_D \sqrt{r^2-x^2}\,dA$ 

Mean Value Inequality

$m \le f(x,y) \le M \implies m\,\text{Area}(D) \le \iint_D f\,dA \le M\,\text{Area}(D)$

"nice"

↳ need integral to exist!

$f: D \to \mathbb{R}$ cont, D elemental, $\exists (x_0,y_0) \in D$ st $\iint_D f\,dA = f(x_0,y_0)\cdot \text{Area}(D)$

Proof: We know fn takes on max/min values, say at $\vec{u}$ and $\vec{l}$

Let $c(t)$ be a cont, diff path from $\vec{l}$ to $\vec{u}$

↳ exists by elemental, often straight line or unions

↳ must make rigorous 

↳ Jordan plane curve thm hard to prove

Consider $h(t) = f(c(t))$, $h(0) = f(\vec{l})$ and $h(1) = \vec{u}$

By IVT, h attains all values b/w $f(\vec{l})$ and $f(\vec{u})$

Note $m = f(\vec{l}) \le \dfrac{\iint_D f\,dA}{\text{Area}(D)} \le f(\vec{u}) = M$

Thus $\exists (x_0,y_0)$ st $f(x_0,y_0) = \iint_D f\,dA / \text{Area}(D)$

↳ and some idea on how to find

Pg 353: HW: #16, #4, #14

Suggested: #6, #15

5.5. THE TRIPLE INTEGRAL

Completely straight forward generalization

Start with rectangles, generalize to elemental

↪ W elemental with z bounded b/w two fns x and y

$$\iiint_W f\,dV = \int_{x=a}^{b} \int_{y=\varphi_1(x)}^{\varphi_2(x)} \int_{z=\gamma_1(x,y)}^{\gamma_2(x,y)} f\,dz\,dy\,dx$$

$$\text{or} \quad = \int_{y=c}^{d} \int_{x=\psi_1(x)}^{\psi_2(x)} \int_{z=\gamma_1(x,y)}^{\gamma_2(x,y)} f\,dz\,dx\,dy$$

HW: pg 363: #3, #9, #12
Suggested #8, #19, #27

SUPPLEMENTAL TOPIC: MONTE CARLO INTEGRATION

History: Most $\int \cdots \int$ too hard to do in closed form (Finance: 360^+ dim)

↳ ex: $\int_a^b e^{-x^2/2}\, dx$

↳ using prob can approx very well with high confidence

Prob Review

- density: $f(x) \geq 0$, $\int_{-\infty}^{\infty} f(x)\, dx = 1$
- mean $\mu = \int_{-\infty}^{\infty} x f(x) dx$ if exists, ave value, write $E[X]$
- variance $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$: measures how spread out $= E[(X-\mu)^2]$

↳ std dev σ, same units as μ

↳ aside k^{th} centered moment $\mu_k = \int_{-\infty}^{\infty} (x-\mu)^k f(x) dx$

↳ like Taylor coeff: know moments, know fn (if nice)

- Chebyshev: $\text{Prob}(|X-\mu| \leq k\sigma) \geq 1 - \frac{1}{k^2}$

↳ Proof: $\text{Prob}(|X-\mu| > k\sigma) = \int_{|X-\mu|>k\sigma} f(x)\, dx$

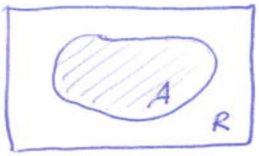
$$\leq \int_{|X-\mu|>k\sigma} \frac{(x-\mu)^2}{k^2\sigma^2} f(x) dx$$

$$\leq \frac{1}{k^2\sigma^2} \int (x-\mu)^2 f(x) dx = \frac{1}{k^2}$$

↳ in general have stronger results, but holds $\forall$ distr with finite mean and variance.

Supplemental Topic: Monte Carlo Integration

Simple case:



DARTS!

"Nice" region A inside rectangle R ($= [0,1]^n$, say)

Assume easy to tell if $\vec{x} \in A$ or $\vec{x} \notin A$

Choose N points unif in R ($= [0,1]^n$), each choice <u>indep</u> of others

↳ Prob a point is in A is $Vol(A)/Vol(R) = Vol(A) \equiv p$.

$$X_i = \begin{cases} 1 & \text{if inside } A \quad \text{happens with prob } p \\ 0 & \text{if } i^{th} \text{ point } \notin A \quad \text{" " " } 1-p. \end{cases}$$

$X = X_1 + \dots + X_N$

What is expected value (mean) of X? What is its std dev?

<u>Lemma</u>: $\mathbb{E}[\Sigma X_i] = \Sigma \mathbb{E}[X_i]$

$Var(\Sigma X_i) = \Sigma Var(X_i)$ if indep

Each X_i same distr: mean is $1 \cdot p + 0 \cdot (1-p) = p$

Var is $(1-p)^2 \cdot p + (0-p)^2 (1-p) = p(1-p) \cdot 1$

So $\mathbb{E}[X] = Np$

$Var(X) = p(1-p)N$ or Std Dev$(X) = \sqrt{p(1-p)}\sqrt{N}$

SUPPLEMENTAL TOPIC: MONTE CARLO INTEGRATION

Let N be enormous

Chebyshev: "high" prob that X very close to Np

↳ Prob at least $1 - \frac{1}{B^2}$ that $|X - Np| \le B \cdot \sqrt{p(1-p)}\sqrt{N}$

Estimate of volume is $\frac{X}{N}$, true vol is p

Thus $\left|\frac{X}{N} - p\right| \le \frac{B\sqrt{p(1-p)}}{\sqrt{N}}$ with prob at least $1 - \frac{1}{B^2}$

↳ max error at $p = 1/2$

↳ "trouble" getting accurate answer if p or $1-p$ very small

Exercises: All supplemental

① Write a program to estimate area of quarter-circle.

Record errors for various N.

② f prob distr with μ, σ finite. Prove $\exists\, a, b$ st

↳ $g(x) = f(x + a)$ has mean 0 and std dev σ

↳ $h(x) = g(bx) = f(b(x+a))$ has mean 0 and var 1

Important: normalizes: easier to compare different distr

↳ "nice" distr: don't see shape until 3rd/4th moment

↳ universality $\Longrightarrow$ CLT (Math 161-162)