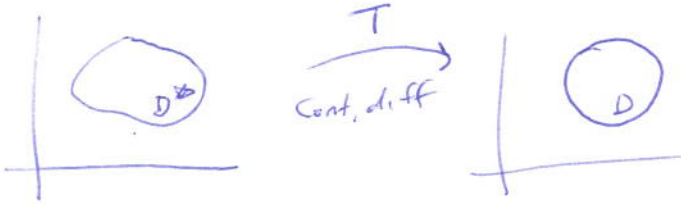
Chapter 6: The Change of Variables Formula + Apps to $\int$

6.1. The Geometry of Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$



ex: $T: [0,1] \times [0, 2\pi] \to$ unit circle : $T(r,\theta) = (r\cos\theta, r\sin\theta)$

↳ not 1-1 / inj: $\{0\} \times [0, 2\pi] \longrightarrow (0,0)$

is onto / surj: all points in D hit

__Thm:__ A 2x2 matrix, Det $A \neq 0$, maps ▱ to ▱, vertices → vertices

__Thm (Adv):__ D^*, D elementary, $T: D^* \to D$ st $DT(u,v) \neq \vec{0}$ $\forall (u,v) \in D^*$,
if $T: \partial D^* \xrightarrow[\text{onto}]{1\text{-}1} \partial D$ Then T is 1-1, onto (bij)

__Thm:__ Det $A \neq 0 \Leftrightarrow A$ is 1-1, onto

HW: pg 375: #2, #7, #12
Suggested: #3, #8

6.2. THE CHANGE OF VARIABLE FORMULA

$T : D^* \to D$ with $T(u,v) = (x,y) = (x(u,v), y(u,v))$

Can't be $\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v))\,du\,dv$

↳ in general D, D^* diff areas, take $f \equiv 1$

Defn: Jacobian Det

$$\text{“}JF\text{''} = DT(u_0,v_0) = \begin{vmatrix} \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial x}{\partial v}(u_0,v_0) \\ \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial v}(u_0,v_0) \end{vmatrix} \overset{\text{def}}{=} \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

Yields $\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$

Think of as $\frac{dx\,dy}{du\,dv}\,du\,dv$

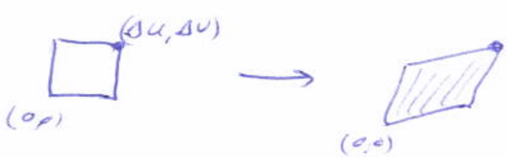
↳ assuming T is C^1, 1-1

Justification

Pg 134: Differentiability $g(\vec{v}) = g(\vec{v}_0) + Dg(\vec{v}_0) \cdot (\vec{v} - \vec{v}_0) + \text{small}$

w/o.g. $(u_0, v_0) = (x_0, y_0) = (0,0)$, $T(u_0,v_0) = (x_0, y_0)$

$$T(\Delta u, \Delta v) \approx \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} + \text{small}$$



so Area $\Delta x \Delta y \approx |\text{Det } DT| \cdot \Delta u \Delta v$

Change of vars: 1-var

$$\int_a^b f(x(u)) \cdot x'(u)\,du = \int_{x(a)}^{x(b)} f(x)\,dx \quad \text{Chain Rule}$$

6.2. The Change of Variables Formula

Polar Coords: $\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta$

↳ not 1-1 everywhere but ok

Ex: Gaussian integral (and $\Gamma(1/2)$ and Wallis! Factorials...)

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$: Square and change to polar

Triple and higher

$$\begin{vmatrix} \partial x_1/\partial u_1 & \cdots & \partial x_1/\partial u_n \\ \vdots & & \vdots \\ \partial x_n/\partial u_1 & \cdots & \partial x_n/\partial u_n \end{vmatrix} = \left|\frac{\partial(x_1,\dots,x_n)}{\partial(u_1,\dots,u_n)}\right|$$

Cylindrical: $r\,dr\,d\theta\,dz$ $\quad x = r\cos\theta,\ y = r\sin\theta,\ z = z$

Spherical: $\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$ $\quad x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi$

HW: pg 390: #2, #3a, #11, #29, #34

Suggested: #4, #7, #10, #15, #18, #31

6.4. IMPROPER INTEGRALS

One variable: $\int_0^1 x^{-1/2}\,dx$ or $\int_1^\infty x^{-2}\,dx$

↳ use limits: $\lim_{\varepsilon\to 0}\int_\varepsilon^1 x^{-1/2}\,dx$

Several vars: look at contraction, expand

$D=\{(x,y): a\le x\le b,\ \varphi_1(x)\le y\le \varphi_2(x)\}$

$D_{\eta,\delta}=\{(x,y): a+\eta\le x\le b-\eta,\ \varphi_1(x)+\delta\le y\le \varphi_2(x)-\delta\}$

$$\lim_{(\eta,\delta)\to(0,0)}\iint_{D_{\eta,\delta}} f\,dA$$

if this exists, then $\lim_{\eta\to 0}\lim_{\delta\to 0}\int_{a+\eta}^{b-\eta}\int_{\varphi_1(x)+\delta}^{\varphi_2(x)-\delta} f\,dA$ exists

However, iterated limits exist $\not\Rightarrow$ $\lim_{(\eta,\delta)\to(0,0)}$ exists

↳ example in book

Fubini: D elemental, $f\ge 0$, $\left[\iint_D f\,dA,\ \int_a^b\int_{\varphi_1(x)}^{\varphi_2(x)} f\,dy\,dx\right.$ for y-simple or $\int_c^d\int_{\psi_1(y)}^{\psi_2(y)} f\,dx\,dy$ for x-simple exist$\Big]$ then f integrable and all equal.

↳ advanced: $\sum\sum a_{mn}\ne\sum\sum a_{nm}$

$$\begin{matrix} 0 & 0 \\ 0 & -1 \\ -1 & +1 \\ +1 & 0 & \cdots \end{matrix}$$

HW: pg 415: #5, #17, #18

Suggested: #1, #12, #13

Gamma Function: $\Gamma(n) = (n-1)!$

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\,dx \quad \left(= 2\int_0^\infty e^{-t^2} t^{2s-1}\,dt \right)$$

$$\Gamma(x)\Gamma(-x) = -\frac{\pi}{x\sin(\pi x)}$$

$$|(ix)!|^2 = \frac{\pi x}{\sinh(\pi x)}$$

$$\Gamma(2x) = (2\pi)^{-1/2}\, 2^{2x-1/2}\, \Gamma(x)\,\Gamma(x+\tfrac{1}{2})$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

Stirling: $\Gamma(z) \sim e^{-z} z^{z-1/2} \sqrt{2\pi}\left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \cdots\right)$

$$\Gamma(\tfrac{s}{2})\,\pi^{-s/2}\,\zeta(s) = \Gamma(\tfrac{1-s}{2})\,\pi^{-(1-s)/2}\,\zeta(1-s)$$

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \frac{(p-1)!(q-1)!}{(p+q-1)!}$$

$$\begin{aligned}
m!\,n! &= \int_0^\infty e^{-u} u^m\,du \int_0^\infty e^{-v} v^n\,dv & u &= x^2,\ v = y^2 \\
&= 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} |x|^{2m+1} |y|^{2n+1}\,dx\,dy & x &= r\cos\theta,\ y = r\sin\theta \\
&= \int_0^{2\pi}\int_0^\infty e^{-r^2} |r\cos\theta|^{2m+1} |r\sin\theta|^{2n+1}\, r\,dr\,d\theta \\
&= \int_0^\infty e^{-r^2} r^{2m+2n+3}\,dr \int_0^{2\pi} |\cos^{2m+1}\theta \sin^{2n+1}\theta|\,d\theta \\
&= 4\int_0^\infty e^{-r^2} r^{2m+2n+3}\,dr \int_0^{\pi/2} \cos^{2m+1}\theta \sin^{2n+1}\theta\,d\theta \\
&= 2(m+n+1)! \int_0^{\pi/2} \cos^{2m+1}\theta \sin^{2n+1}\theta\,d\theta & u &= \cos^2\theta
\end{aligned}$$

$$B(m+1, n+1) = \frac{m!\,n!}{(m+n+1)!} = \int_0^1 u^m (1-u)^n\,du$$

↳ can generalize to non-integers

↳ u-integrals easy to bound

↳ use: $\left|\frac{\Gamma(a \pm iy)}{\Gamma(b \pm iy)}\right| = O_{a,b}(1)$ — pos signs: $\frac{\Gamma(a+iy)\Gamma(b-a)}{\Gamma(b+iy)} = \int_0^1 t^{a+iy-1}(1+t)^{b-a-1}\,dt$

↳ arises in shifts of L-fns

GAMMA AND BETA FUNCTIONS + DISTRIBUTIONS

GAMMA FUNCTION

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy \quad \text{for } Re(\alpha) > 0$$

- NOTE: $\Gamma(1) = 1$ and $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ for $Re(\alpha) > 0$

$\hookrightarrow \Gamma(1) = \int_0^\infty e^{-y}\, dy = 1$

$\Gamma(\alpha+1) = \int_0^\infty y^{\alpha} e^{-y}\, dy$ integrate by parts:

$$\begin{cases} u = y^\alpha & dv = e^{-y}\, dy \\ du = \alpha y^{\alpha-1} & v = -e^{-y} \end{cases}$$

$$= -y^\alpha e^{-y}\Big|_0^\infty + \int_0^\infty \alpha y^{\alpha-1} e^{-y}\, dy$$

$$= \alpha \int_0^\infty y^{\alpha-1} e^{-y}\, dy$$

$$= \alpha\,\Gamma(\alpha)$$

- NOTE: If $\alpha = n$ (a positive integer) Then $\Gamma(n+1) = n!$
  This follows from repeated use of $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ and $\Gamma(1) = 1$

THM: $\Gamma(1/2) = \sqrt{\pi}$

we shall use this technique again in studying the Beta distribution.
Let $a \in (0,1)$ so $\Gamma(a)$, $\Gamma(1-a)$ exist and are finite. As all integrands below are positive, all change of variables can be justified.

$$\Gamma(a)\Gamma(1-a) = \int_0^\infty\int_0^\infty x^{a-1} e^{-x}\, y^{1-a-1} e^{-y}\, dx\, dy$$

$$= \int_0^\infty\int_0^\infty \left(\frac{x}{y}\right)^{a-1} e^{-\left(\frac{x}{y}\right)y}\, y^{-1} e^{-y}\, dx\, dy$$

Change variables. Fix y, set $z = \frac{x}{y}$, $dz = \frac{dx}{y} = y^{-1}dx$

$$= \int_0^\infty\int_0^\infty z^{a-1} e^{-zy} e^{-y}\, dz\, dy$$

$$= \int_0^\infty\int_0^\infty z^{a-1} e^{-(z+1)y}\, dz\, dy \quad \begin{cases}\text{Change Vars: } u = (z+1)y, \ du = (z+1)\,dy \\ \text{Fix } z\end{cases}$$

$$= \int_0^\infty\int_0^\infty z^{a-1} e^{-u} \frac{du}{z+1}\, dz = \int_0^\infty \frac{z^{a-1}}{z+1} \underbrace{\int_0^\infty e^{-u}\, du}_{=\Gamma(1)=1}\, dz$$

$$= \int_0^\infty \frac{z^{a-1}}{1+z}\, dz \quad \text{Change vars: } z = u^2,\ dz = 2u\, du$$

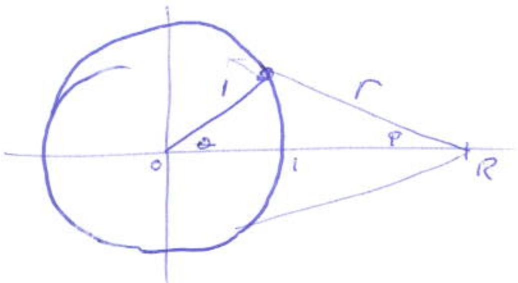
$$= \int_0^\infty u^{2a-2}/(1+u^2) \cdot 2u\, du \quad \text{Change vars: } u = \tan\theta,\ du = \sec^2\theta\, d\theta$$

$1 + \tan^2\theta = \sec^2\theta$, $u = 0 \to \theta = 0$, $u = \infty \to \theta = \pi/2$

$$= 2\int_0^{\pi/2} \frac{(\tan\theta)^{2a-1}}{\sec^2\theta} \sec^2\theta\, d\theta : \text{If } a = 1/2 \text{ get } 2\int_0^{\pi/2} d\theta = \pi$$

6.3. Applications

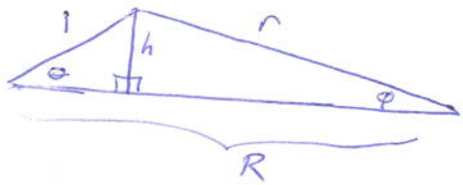
Average values / center of mass: $\bar{x}_i = \dfrac{\int\cdots\int x_i\, \delta(x_1,\dots,x_n)\,dx_1\cdots dx_n}{\int\cdots\int \delta(x_1,\dots,x_n)\,dx_1\cdots dx_n}$



$$r^2 = R^2+1-2R\cos\theta$$

Force from point: $G\dfrac{m_1m_2}{r^2}\hat{e}_r$

Force $G\dfrac{m_1m_2}{r^2}\cos\varphi\cdot(-\hat{e}_x)$



$$1 = R^2+r^2-2Rr\cos\varphi$$

So $\cos\varphi = \dfrac{R^2+r^2-1}{2Rr}$

So $\cos\varphi = \dfrac{2R^2-2R\cos\theta}{2Rr} = \dfrac{\cancel{R}}{r}(R-\cos\theta)$

Force $= Gm_1m_2\cancel{R}\dfrac{R-\cos\theta}{(R^2+1-2R\cos\theta)^{3/2}}$

Force $= Gm_1m_2\cancel{R}(R-\cos\theta)\dfrac{\sin^3\varphi}{\sin^3\theta}$

Force $= Gm_1m_2\cancel{R}\dfrac{R-x}{(R^2+1-2R\cancel{x})^{3/2}} = Gm_1m_2\dfrac{R-x}{\ }$

$\dfrac{r}{\sin\theta} = \dfrac{1}{\sin\varphi}$

$r\sin\varphi = \sin\theta$

$\dfrac{1}{r} = \dfrac{\sin\varphi}{\sin\theta}$

$\sin\theta = r\sin\varphi$

$r^2 = \dfrac{\cancel{R^2}(R-\cos\theta)^2}{\cos^2\varphi}$

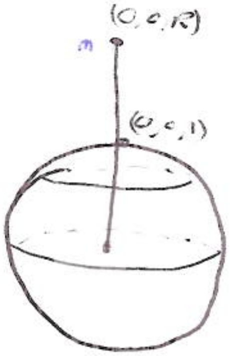
$r = \dfrac{\cancel{R}(R-\cos\theta)}{\cos\varphi}$

Do in circular slices

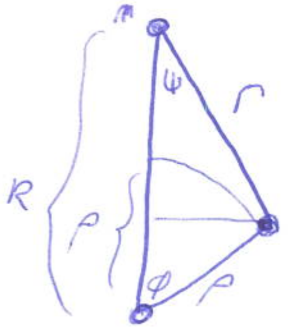
↳ Given θ, mass $2\pi\sin\theta\cdot Gm\dfrac{\cos\varphi}{r^2} = 2\pi Gm\dfrac{\sin\varphi\cos\varphi}{r}$

Easier to do volumes! Careful w/ arc-length

6.3, APPLICATIONS



Assume unit density of 1, total mass $\frac{4}{3}\pi$



$\cos\psi = \dfrac{R-\rho\cos\varphi}{r}$ $\quad r^2 = R^2+\rho^2-2R\rho\cos\varphi$

Still true if $\varphi > \pi/2$

F in $\hat{e}_r$ dir, amount in z-dir is $F\cos\psi$

$$\text{Force} = \iiint F(x,y,z)\cos\psi\, dx\,dy\,dz = \iiint F(\rho,\theta,\varphi)\cos\psi\, \rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi$$

$$= Gm\iiint \frac{R-\rho\cos\varphi}{(R^2+\rho^2-2R\rho\cos\varphi)^{3/2}}\rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi$$

$$= 2\pi Gm\int_{\rho=0}^{1}\frac{\rho}{R}\int_{\varphi=0}^{\pi}(R-\rho\cos\varphi)\cdot(R^2+\rho^2-2R\rho\cos\varphi)^{-\frac{3}{2}}R\rho\sin\varphi\, d\varphi\, d\rho$$

$u = R-\rho\cos\varphi \qquad dv = (R^2+\rho^2-2R\rho\cos\varphi)^{-\frac{3}{2}}R\rho\sin\varphi\, d\varphi$

$du = \rho\sin\varphi\, d\varphi \qquad v = -(R^2+\rho^2-2R\rho\cos\varphi)^{-\frac{1}{2}}$

Note $uv\Big|_0^{\pi} = \frac{R-\rho}{R-\rho} - \frac{R+\rho}{R+\rho} = 0$

$$= 2\pi Gm\int_{\rho=0}^{1}\frac{\rho}{R}\frac{1}{R}\int_{\varphi=0}^{\pi}(R^2+\rho^2-2R\rho\cos\varphi)^{-\frac{1}{2}}\cdot R\rho\sin\varphi\, d\varphi$$

$$= \frac{2\pi Gm}{R^2}\int_{\rho=0}^{1}\rho\left[(R+\rho)-(R-\rho)\right]d\rho$$

$= Gm(\frac{4}{3}\pi)/R^2$ as ρ-integral is $\int 2\rho^2 d\rho = \frac{2}{3}$

See potential formulation in book is easier! <u>Keep track of units</u>

HW: Pg 404: #1

Suggested: #3, #17, #18

Extra Credit: $F \propto 1/r^n$ in n-dim space: concentrated in center

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HW: pg 415: #5, #17, #18

Suggested: #1, #12, #13