CHAPTER 7: INTEGRALS OVER PATHS AND SURFACES

7.1. THE PATH INTEGRAL

Defn: Path integral ($\int$ of f along the path c): $c:[a,b] \to \mathbb{R}^n$ of class C^1,
$f(c(t))$ cont on I, then
$$\int_c f\,ds = \int_a^b f(x_1(t),\dots,x_n(t)) \cdot \|c'(t)\|\,dt = \int_a^b f(c(t))\,\|c'(t)\|\,dt$$
(if only piecewise cont, break into stages) (Think of as density)

Ex: $c(t) = (\sin t, \cos t, t)$ $\|c'(t)\| = \sqrt{2}$
$f(x,y,z) = \cos z$, $f(c(t)) = \cos t$, $\int_c f\,ds = \int_0^{2\pi} \cos t \cdot \sqrt{2}\,dt = 0$

HW: pg 427: #3a, #6, #13
Suggested: #4a, #8, #12, #16

7.2. Line Integrals

Force * displacement in dir of force = work = $\int_a^b F(c(t))\cdot c'(t)\,dt$

↳ subdivide as always, take limit

$\xrightarrow[\text{Force}]{}$ ↑ path no work done by Force   Ex: 

line integral of F along C (C is C^1, F is C^1): $\int_c F\cdot ds = \int_a^b F(c(t))\cdot c'(t)\,dt$

↳ similar defn if piecewise C^1

$T(t) = c'(t)/\|c'(t)\|$ if $\|c'(t)\|$ never 0: unit speed parametrization

$\int_c F\cdot ds = \int_a^b F(c(t))\cdot T(t)\,\|c'(t)\|\,dt$

Often write $\int_c F\cdot ds = \int_c \underbrace{F_1\,dx_1 + \cdots + F_n\,dx_n}_{\text{call this a differential form}}$

$ds = \hat{e}_1\,dx_1 + \cdots + \hat{e}_n\,dx_n$

$F\cdot ds$ is dot product

defn: $c(t) = (x_1(t), \dots, x_n(t))$ then

$$\int_c F_1dx_1+\cdots+F_ndx_n = \int_a^b \left(F_1 \frac{dx}{dt}\right)dt + \cdots + \left(F_n\frac{dx_n}{dt}\right)dt$$

Ex: $\int_c x^2dx + xy\,dy + dz$, $c(t) = (t, t^2, 1)$ $0\le t\le 1$

$dx/dt = 1$ $dy/dt = 2t$ $dz/dt = 0$ (no Δ in z-dir)

$= \int_0^1 t^2\,dt + 2t^4\,dt$

Reparametrizations

↳ trace out same path in space (possibly opposite order)

↳ same # of times (ex: $(\cos t, \sin t)$ and $(\cos 2t, \sin 2t)$ are not same if both from $t=0 \to 2\pi$)

$c: I' \to \mathbb{R}^n$ ~~$C^1: I' \to \mathbb{R}^n$~~ C^1, 1-1 $h: I \to I'$

Then $p = c\circ h: I\to\mathbb{R}^n$ a reparametrization of c

↳ changes speed if orientation preserving

7.2. Line Integrals

h a reparametrization $I = [a,b]$ $\quad I' = [a',b']$ $\quad c: I' \to \mathbb{R}^n$

↳ orientation preserving if $c(h(a)) = c(a')$, $c(h(b)) = c(b')$

reversing if $\quad = c(b') \quad\quad = c(a')$

<u>Thm</u>: cont F on C^1 path $c: [a',b'] \to \mathbb{R}^n$, p reparam of c

$$\int_p F \cdot ds = \text{sign}(p) \int_c F \cdot ds$$

where $\text{sign}(p) = \begin{cases} +1 & \text{orientation preserving} \\ -1 & \text{reversing} \end{cases}$

> Extremely important: answer should be indep of how param

<u>Proof</u>: $p = c \circ h$ so $p'(t) = c'(h(t))\, h'(t)$

$$\int_p F \cdot ds = \int_a^b F(c(h(t))) \cdot c'(h(t))\, h'(t)\, dt$$

let $u = h(t)$

$$\int_p F \cdot ds = \int_{h(a)}^{h(b)} F(c(u)) \cdot c'(u)\, du = \text{sign}(p) \int_c F \cdot ds$$

from whether or not endpoints of $\int$ in right order

Line Integrals of Gradient Fields

F gradient field if $\exists f$ st $F = \nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$

One-var: $G' = g$ then FTC: $\int_a^b g(x)\,dx = G(b) - G(a)$

↳ only depends on endpoints!

Generalization: $\int_c \nabla f \cdot ds = f(c(b)) - f(c(a))$

Proof: $F(t) = f(c(t))$ then $F'(t) = \nabla f(c(t)) \cdot c'(t)$

7.2. LINE INTEGRALS

<u>Simple curve</u>: C image piecewise $\mathcal{C}^1$ map $c: I \to \mathbb{R}^n$

call c a param of C, endpoints $c(a)$, $c(b)$

Each curve has two orientations 

↳ specify orientation, call it <u>oriented</u> (or <u>directed</u>) <u>simple curve</u>

<u>Simple closed curve</u> same except 1-1 on $[a,b)$ and $c(a)=c(b)$

C oriented simple curve (or closed curve), c orientation preserving param

$$\int_C F\cdot ds = \int_c F\cdot ds \qquad \int_C f\,ds = \int_c f\,ds$$

$$\int_C F\cdot ds = -\int_{C^-} F\cdot ds \quad (C^- \text{ opposite orientation})$$

$$C = C_1 + \cdots + C_k$$  $$\int_C F\cdot ds = \sum_{i=1}^{k} \int_{C_i} F\cdot ds$$

<u>dr Notation</u>

Often write instead $\int_C F\cdot dr$

↳ Think of describing $\mathcal{C}^1$ path c in terms of a moving position vector based at origin and ending at $c(t)$ at time t.

denote position vectors by $\vec{r}(t) = (x_1(t), \ldots, x_n(t))$

$$\int_c F\cdot dr = \int_a^b F(r(t))\cdot \frac{dr}{dt}\,dt$$

HW: pg 447: #1ad, #4, #11, #14

Suggested: #3, #5, #13, #17

7.3. Parametrized Surfaces

Graphs too restrictive: forced to be ~~ in 

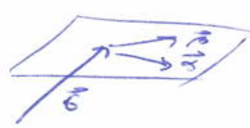
Defn: Parametrized Surface

$\Phi : D \to \mathbb{R}^3 \quad D \subset \mathbb{R}^2$, surface S is image $\Phi(D)$

$\Phi(u,v) = (x(u,v), \ldots, z(u,v))$

Φ diff or C^1 (same as each coord fn diff or C^1) say

S is a diff or C^1 surface

Ex: Plane:  $\Phi(u,v) = u\vec{\alpha} + v\vec{\beta} + \vec{\gamma}$

Tangent Vectors to Param surfaces

Compare to what we had with Jacobian in change of var formula

Φ param surf, diff at (u_0,v_0)

map $t \mapsto \Phi(u_0, t)$ is a curve on surface

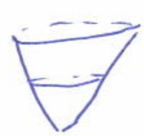
tangent at (u_0,v_0) to curve is $T_u = \frac{\partial \Phi}{\partial v} = \frac{\partial x}{\partial v}(u_0,v_0)\hat{i} + \cdots + \frac{\partial z}{\partial v}(u_0,v_0)\hat{k}$

Similarly $t \mapsto \Phi(t, v_0)$ yields $T_v = \frac{\partial \Phi}{\partial u} = \frac{\partial x}{\partial u}(u_0,v_0)\hat{i} + \cdots + \frac{\partial z}{\partial u}(u_0,v_0)\hat{k}$

Thus $T_u \times T_v$ is normal to surface (only dir left)

↳ advantage of maps from $\mathbb{R}^2 \to \mathbb{R}^3$

Defn: surface S regular or smooth at $\Phi(u_0,v_0)$ if $T_u \times T_v \neq \vec{0}$ at (u_0,v_0). Called regular if regular everywhere.

Ex: $T(u,v) = (u\cos v, u\sin v, u) \quad u \geq 0$  $z = \sqrt{x^2+y^2}$ cone

↳ diff surface as each coord fn diff

↳ not regular as $T_v = \vec{0}$ at $(0,0)$

Defn: Tangent Plane: $T_u \times T_v \neq \vec{0}$ at (u_0,v_0), $\Phi(u_0,v_0) = (x_0,y_0,z_0)$, then tangent plane is $(x-x_0, y-y_0, z-z_0)\cdot \vec{n}$ $\quad \vec{n} = T_u \times T_v$

HW: pg 459: #1, #9a, #16

Suggested: #4, #10, #13, #14, #17, #18

7.4. AREA OF A SURFACE

$\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$ param surface S

regular at (u_0,v_0) if $T_u = \frac{\partial x}{\partial u}(u_0,v_0)\vec{e}_1 + \dots + \frac{\partial z}{\partial u}(u_0,v_0)\vec{e}_3$

$T_v = \frac{\partial x}{\partial v}(u_0,v_0)\vec{e}_1 + \dots + \frac{\partial z}{\partial v}(u_0,v_0)\vec{e}_3$

are st $T_u \times T_v \neq \vec{0}$

Consider piecewise regular surfaces that are unions of images of param surfaces $\Phi_i : D_i \to \mathbb{R}^3$ st

(1) D_i elemental in $\mathbb{R}^2$

(2) Φ_i class C^1, 1-1 except on ∂D_i

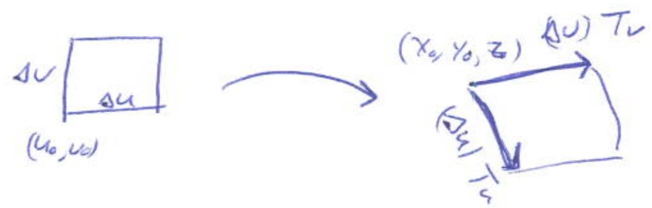
(3) $S_i = \Phi_i(D_i)$ is regular except at finitely many points

Defn: Surface Area $A(S)$ of param surface (S, Φ):

$$A(S) = \iint_D \|T_u \times T_v\| \, du\, dv$$

↳ show indep of param

Justification:



So area $\Delta u \Delta v \rightsquigarrow \|T_u \times T_v\| \Delta u \Delta v$

Formulas: $\|T_u \times T_v\| = \left(\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(u,v)}\right)^2 \right)^{1/2}$

where $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$ ...

Direct Calculation: ex: $\Phi(u,v) = (r\cos\theta, r\sin\theta, r)$

Then $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

7.4. AREA OF A SURFACE

Surface area of a graph:

$\Phi(u,v) = (u, v, g(u,v))$

$$A(S) = \iint_D \sqrt{\left(\partial g/\partial x\right)^2 + \left(\partial g/\partial y\right)^2 + 1} \; dx\,dy$$

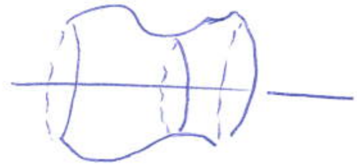
as $x = u$, $\partial g/\partial u = \partial g/\partial x \ldots$

straight forward

Surface of Revolution

1-var: rotate about x-axis: $A = 2\pi \int_a^b |f(x)| \sqrt{1 + |f'(x)|^2} \, dx$

↳ can derive @ our results



$\Phi(u,v) = (u, f(u)\cos v, f(u)\sin v)$

$a \le u \le b \qquad 0 \le v \le 2\pi$

Follows by direct calc: $\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2 = f(u)^2 \sin^2 v, \ldots$

HW: pg 471: #1, #2, #9, #16, #23

Suggested: #3, #8, #10, #14, #18, #21

7.5. INTEGRALS OF SCALAR FNS OVER SURFACES

Φ param surface S, f cont on S, integral of f over S is

$$\iint_S f(x,y,z)\,dS = \iint_S f\,dS = \iint_D f(\Phi(u,v))\,\|T_u \times T_v\|\,du\,dv$$

↳ same argument as always

↳ f=1 reduces to surface area

$$\iint_S f\,dS = \iint_D f(\Phi(u,v))\sqrt{\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2 + \cdots + \left(\frac{\partial(x,z)}{\partial(u,v)}\right)^2}\,du\,dv$$

$$= \iint_D f(x,y,g(x,y))\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\,dx\,dy$$

↳ if $\Phi(u,v) = (u,v,g(u,v))$

$$= \iint_D f(x,y,g(x,y))\,\frac{dx\,dy}{\cos\theta}$$

↳ $\Phi(u,v) = (u,v,g(u,v))$

surface given by $\phi(x,y,z) = z - g(x,y) = 0$

↳ normal $= \nabla\phi = -\frac{\partial g}{\partial x}\hat{i} - \frac{\partial g}{\partial y}\hat{j} + \hat{k} = \vec{N}$

$$\cos\theta = \frac{\vec{N}\cdot\hat{k}}{\|\vec{N}\|} = 1\Big/\sqrt{(\partial g/\partial x)^2 + (\partial g/\partial y)^2 + 1}$$

HW: pg 480: #1, #6, #11ab, #17

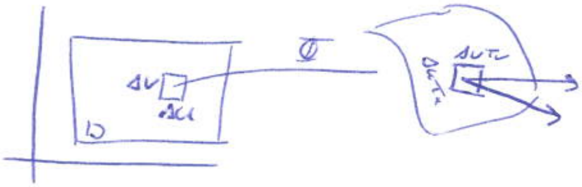
Suggested: #5, #7, #9, #15, #16

7.6. Surface Integrals of Vector Fields

F vector field, S surface param Φ. The surface $\int$ of F over Φ is

$$\iint_\Phi F\cdot d\vec{S} := \iint_D F\cdot(T_u\times T_v)\,du\,dv$$

Physical interpretation: flux thru surface!



if $F \perp T_u\times T_v$, nothing leaves

$\boxed{\text{Feynman: why gravity is a } 1/r^2 \text{ force}}$

Oriented surface: 2 sided surface w/ one side specified as outside (pos) and other specified as inside (neg). At each point two normals: $\vec{n}_1$, $\vec{n}_2 = -\vec{n}_1$

Not every surface orientable:  Möbius

$T_u\times T_v / \|T_u\times T_v\| = \pm n(\Phi(u,v))$ with n the unit normal

↳ orientation preserving if +

reversing if −

$$\iint_S F\cdot dS = \operatorname{sign}(\Phi)\iint_\Phi F\cdot dS \leftarrow \text{indep (up to sign) on } \Phi$$

$$= \iint_S F\cdot n\, dS \qquad dS = \|T_u\times T_v\|\,du\,dv$$

Let $F = c\vec{r}$, S unit sphere with outward normal

$$\iint_S F\cdot dS = \iint_S F\cdot n\, dS = c\iint_S dS = \tfrac{4}{3}\pi c$$

See book for additional formulas

See pg 496 for summary

HW: pg 497: #3, #8, #11

Suggested: #5, #14, #15