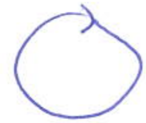
CHAPTER 8: THE INTEGRAL THMS OF VECTOR CALCULUS

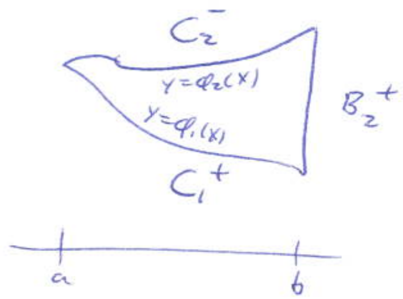
8.1. GREEN'S THM

pos orientation
counter-clockwise



neg orientation
clockwise

boundary y-simple: top, bottom (and possibly sides)



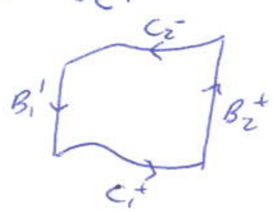
+ means pos
− means neg

<u>Greens Thm</u>: D simple @ $\partial D = C$, $P, Q : D \to \mathbb{R}$ of C^1, then

$$\int_{C^+} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\int_{\partial D} \omega = \int_D d\omega$$

<u>Lemma</u>: $\int_{C^+} P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dx\,dy$



Fubini:

$$\iint_D \frac{\partial P}{\partial y}(x,y)\,dx\,dy = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y}(x,y)\,dy\,dx$$

$$= \int_a^b \left[P(x,\varphi_2(x)) - P(x,\varphi_1(x))\right] dx$$

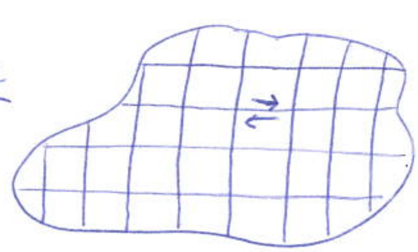
$$= -\int_{C_2^-} P(x,y)\,dx - \int_{C_1^+} P(x,y)\,dx$$

integrals on B_1^+, B_2^+ vanish as $dx=0$ there

<u>Lemma</u>: $\int_{C^+} Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dx\,dy$

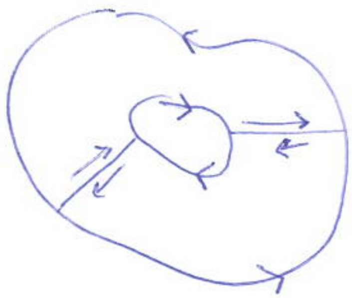
↳ can generalize to regions

<u>Alternate</u>



notice internal pieces cancel
difficulty only on boundary

8.1. GREEN'S THM



Notice orientation of inner region

Prove Green's in each region, add

Application: Area

C simple closed curve, bounds region where Green applies $(\partial D = C)$

$$A = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx$$

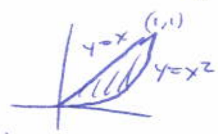
↳ Proof: $P(x,y) = -y \quad Q(x,y) = x$

$\boxed{\begin{array}{ll} \text{examples:} & \text{Do we need} \\ \hookrightarrow \text{circle} & \text{to know area} \\ \hookrightarrow \text{ellipse} & \text{of circle?} \end{array}}$

Thm: Vector form of Green's Thm: $D \subset \mathbb{R}^2$, Green applies, boundary ∂D, $F = P\vec{i} + Q\vec{j}$ is $\mathcal{C}^1$, Then $\int_{\partial D} F \cdot ds = \iint_D (\text{curl } F) \cdot \hat{k}\, dA = \iint_D (\nabla \times F) \cdot \hat{k}\, dA$

Proof: $(\nabla \times F) \cdot \hat{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ when $F = P\vec{i} + Q\vec{j} + 0\hat{k}$

Ex: $F = (xy^2, y+x)$ on



$$\iint_D (\nabla \times F) \cdot k\, dx\,dy = \int_0^1 \int_{x^2}^{x} (1-2xy)\,dy\,dx = \frac{1}{12}$$

$\int_0^1 F_1\,dx + F_2\,dy = \int_0^1 (x^3 + 2x)\,dx = 5/4$
(on curve $y=x$)

$\int_0^1 F_1\,dx + F_2\,dy = \int_0^1 x^5\,dx + (x+x^2)(2x\,dx) = 4/3$
(on curve $y=x^2$)

Thm: Div Thm in Plane: $\vec{n} = \frac{(y'(t), -x'(t))}{(x'(t)^2 + y'(t)^2)^{1/2}}$, $F = P\vec{i} + Q\vec{j}$ is $\mathcal{C}^1$

Then $\int_{\partial D} F \cdot n\, ds = \iint_D \text{div} F\, dA$

$c'(t) = (x'(t), y'(t))$ tangent to ∂D, inspection shows $c'(t) \cdot n = 0$

Calculation: $\int_{\partial D} F \cdot n\, ds = \int_a^b \frac{P(x(t),y(t))\,y'(t) - Q(x(t),y(t))\,x'(t)}{(x'(t)^2 + y'(t)^2)^{1/2}} \sqrt{x'(t)^2 + y'(t)^2}\, dt$

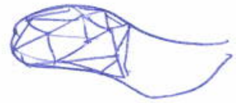
HW: Pg 528: #1, #5, ~~#8~~, #11, #12, #16

Suggested: #2, #6, #8, #15, #18, #19, #21 to #27

8.2. STOKE'S THM

Stoke's Thm: S' oriented surface given by C^2 fn $z = f(x,y)$, $x, y \in D$, Green's Thm applicable in D, F C^1 vector field, $\partial S'$ oriented boundary of S',

$$\iint_{S'} (\text{curl } F) \cdot dS = \iint_{S} (\nabla \times F) \cdot dS' = \int_{\partial S'} F \cdot ds$$

Advanced proof:  triangulate, apply Green's Thm on unde-triangles, ⊕ in ε of above ...

Proof: §7.6: $\iint_{S'} F \cdot dS' = \iint_D \left[F_1 \cdot \left(-\frac{\partial z}{\partial x}\right) + F_2 \cdot \left(-\frac{\partial z}{\partial y}\right) + F_3 \cdot 1\right] dx\,dy$

$$\nabla \times F = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_3}{\partial y}\right)\hat{k}$$

$$(\nabla \times F) \cdot dS = \left[(\nabla \times F)_{\hat{i}} \left(-\frac{\partial z}{\partial x}\right) + (\nabla \times F)_{\hat{j}} \left(-\frac{\partial z}{\partial y}\right) + (\nabla \times F)_{k}\, 1\right] dA$$

Also $\int_{\partial S'} F \cdot ds = \int_p F \cdot ds = \int_p F_1 dx + F_2 dy + F_3 dz = \int_p \left(F_1 \frac{dy}{dt} + \dots\right) dt$

where $p : [a,b] \to \mathbb{R}^3$ $\quad p(t) = (x(t), y(t), f(x(t), y(t)))$

orientation preserving param of oriented simple closed curve $\partial S'$

Chain rule: $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$

Algebra: $\int_{\partial S} F \cdot ds = \int_{\partial D} \left(F_1 + F_3 \frac{\partial z}{\partial x}\right) dx + \left(F_2 + F_3 \frac{\partial z}{\partial y}\right) dy$

Apply Green's Thm, $= \iint_D \left[\frac{\partial (F_2 + F_3 \partial z/\partial y)}{\partial x} - \frac{\partial (F_1 + F_3 \partial z/\partial x)}{\partial y}\right] dA$

Claim follows by algebra + equality of mixed partials

↳ Note: $F_2 = F_2(x, y, z(x,y))$

$$\Rightarrow \frac{\partial F_2}{\partial x} = \frac{\partial F_2}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F_2}{\partial z}\frac{\partial z}{\partial x}$$

proof: $g_y(x) = (x, y, z(x,y))$

Apply chain rule to $F_2(g_y(x))$

8.2. STOKE'S THM

Generalize to surfaces that are not graphs of fns

Warning: $\Phi : D \to S$, $\partial D \not\to \partial S$

↳ pg 357: param sphere

Thm: Stoke's Thm for Param Surfaces

S oriented surface, 1-1 param $\Phi : D \to S$, Green's Thm applies to D, ∂S oriented boundary (may be $\emptyset$), F is C^1. Then

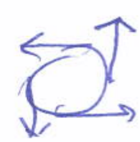
$$\iint_S (\nabla \times F) \cdot dS = \int_{\partial S} F \cdot ds$$

Ex: Sphere:  no boundary: do in 2 pieces, opp orientation so line integrals cancel!

Ex: unit sphere, $F = \vec{r}$. Then $\iint_S (\nabla \times F) \cdot dS = 0$

Physical interpretation of curl

V velocity fluid

$\int_C V \cdot ds$: say V always tangent to oriented curve 

Then $\int_C V \cdot ds > 0$, rotates counter-clockwise

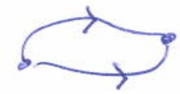
Skim stuff in books on other coord systems

HW: pg 547: #1, #14 (don't do phys interpretation), #17, #23

Suggested: #5, #6, #7, #8, #15, #18, #20, #22

8.3. CONSERVATIVE FIELDS

If $F=\nabla f$ then $\int_c F\cdot ds = f(c(b)) - f(c(a)) = 0$ if closed path

↳ closed ⇒ integral path-indep 

Thm: Conservative Fields

F is C^1 on $\mathbb{R}^3$ except possibly finitely many points. TFAE

(1) C oriented simple closed curve, $\int_C F\cdot ds = 0$

(2) C_1, C_2 " " curves @ same endpoints: $\int_{C_1} F\cdot ds = \int_{C_2} F\cdot ds$

(3) $F = \nabla f$ (f undefined where F undef)

(4) $\nabla \times F = 0$

call such F a conservative vector field.

Proof: (1) → (2) → (3) → (4) → (1)

(1)→(2) Assume $C' = C_1' - C_2'$ simple: $0 = \int_{C'} F\cdot ds = \int_{C_1'} F\cdot ds - \int_{C_2'} F\cdot ds$

↳ if C' not simple, add arguments needed

(2)→(3) Define $f(x,y,z) = \int_C F\cdot ds$ (any path $(0,0,0)$ to (x,y,z)) (or any point F regular)

Choose $C': (0,0,0) \to (x,0,0) \to (x,y,0) \to (x,y,z)$

$$f(x,y,z) = \int_0^x F_1(t,0,0)\,dt + \int_0^y F_2(x,t,0)\,dt + \int_0^z F_3(x,y,t)\,dt$$

FTC: $\frac{\partial f}{\partial z} = F_3(x,y,z)$

other paths give $\partial f/\partial x = F_1$, $\partial f/\partial y = F_2$

(3)→(4) $\forall f$, $\nabla \times \nabla f = 0$

(4)→(1) C represents closed curve C_1, S' surface @ $\partial S' = C$ and if F undefined at points, S' avoids (can do, but must prove).

Stokes: $\int_C F\cdot ds = \int_c F\cdot ds = \iint_{S'} (\nabla \times F)\cdot n \, dS' = 0$

Different in plane (cannot avoid points!)

8.3. Conservative Fields

Example: $F(x,y,z) = y\hat{i} + (z\cos yz + x)\hat{j} + (y\cos yz)\hat{k}$. Find f s.t. $\nabla f = F$

↳ calc: $\nabla \times F = 0$, so f exists

Soln 1: $f(x,y,z) = \int_0^x F_1(t,0,0)\,dt + \int_0^y F_2(x,t,0)\,dt + \int_0^z F_3(x,y,t)\,dt$

$= xy + \sin yz$

Soln 2: f exists: $\frac{\partial f}{\partial x} = y$, $\frac{\partial f}{\partial y} = x + z\cos yz$, $\frac{\partial f}{\partial z} = y\cos yz$

equiv: $f = xy + h_1(y,z)$

$f = xy + \sin yz + h_2(x,z)$

$f = \sin yz + h_3(x,y)$

↳ can match up

↳ $\partial f/\partial z = y\cos yz \Rightarrow \partial h_1/\partial z = y\cos yz \to h_1(y,z) = \sin yz + g(y)$

Planar Case: F is C^1 with "$\nabla \times F$" $= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ then $F = \nabla f$

HW: pg558: #1, #2, #3, #12 (important), #13ac, #16

Suggested: #4, #5, #6, #11, #15, #17, #25

8.4, Gauss' Thm

Review elemental regions: Figs 5.5.2, 5.5.4

Closed surfaces: 2 orientations: outward (pos), inward (neg)

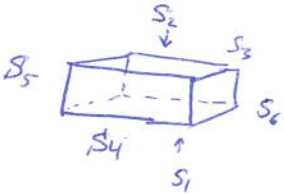
Surface S closed, $S = \sum_{i=1}^{l} S_i \quad 2 \le l \le 6$ elemental

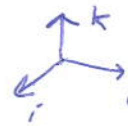
<u>Defn</u>: $\iint_S F \cdot dS = \sum_{i=1}^{l} \iint_{S_i} F \cdot dS$

W closed region, ∂W boundary w/ outward normal, ∂W_{op} w/ inward

$$\iint_{\partial W} F \cdot dS = \iint_S (F \cdot n)\, dS = -\iint_S (F \cdot (-n))\, dS = -\iint_{\partial W_{op}} F \cdot dS$$

Box w/ sides // coord axes:





$F = (F_1, F_2, F_3)$ const

$$\iint_{\partial W} F \cdot dS =$$

$$= \iint_{S_4} F_1\, dS - \iint_{S_3} F_1\, dS + \iint_{S_6} F_2\, dS - \iint_{S_5} F_2\, dS + \iint_{S_2} F_3\, dS - \iint_{S_1} F_3\, dS$$

<u>Thm: Gauss' Divergence Thm</u>

W "nice" region (broken into symmetric elementary regions)

∂W oriented closed boundary surface, F is C^1. Then

$$\iiint_W (\nabla \cdot F)\, dV = \iint_{\partial W} (F \cdot n)\, dS$$

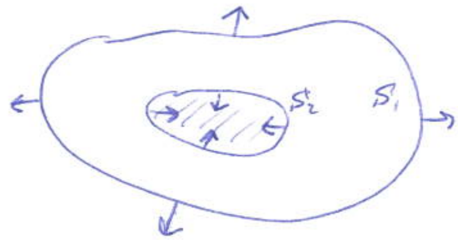
<u>Proof</u>: See book for more general proof: Divide into rectangular boxes, ε-argument to amalgamate

$$\iiint_W \frac{\partial F_1}{\partial x}\, dx\,dy\,dz \underset{\text{Fubini}}{=} \iint \left[\frac{\partial F_1}{\partial x}\, dx\right] dy\,dz = \iint_{S_4} F_1\, dS - \iint_{S_3} F_1\, dS$$

minus sign from orientation <u>and</u> FTC: $\int \frac{\partial F_1}{\partial x}\, dx = F_1(\cancel{x}, y, z) - F_1(0, y, z)$ say

8.4. Gauss' Thm

Can generalize



Two boundary surfaces, S_1 and S_2

Note orientations

Physical interpretation: net outward flux per unit vol: $\nabla \cdot F$

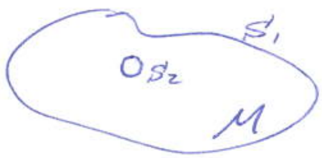
Gauss' Law: M symm elementary region in $\mathbb{R}^3$ $\quad \vec{r} = (x, y, z)$

$$\iint_{\partial M} \frac{\vec{r} \cdot n}{r^3}\, dS = \begin{cases} 4\pi & \vec{0} \in M \\ 0 & \vec{0} \notin M \end{cases}$$

Proof: $\vec{0} \notin M$: $\vec{r}/r^3$ is C^1, $\nabla \cdot (\vec{r}/r^3) = 0$, follows by Div Thm

$\vec{0} \in M$: Draw a sphere about $\vec{0}$ of suff small radius ε st

$B_\varepsilon(\vec{0}) \subset M - \partial M$



As $\nabla \cdot (\vec{r}/r^3) = 0$ in $M - B_\varepsilon(\vec{0})$,

$$\iint_{\underset{S_1}{\overset{\shortparallel}{\partial M}}} \frac{\vec{r} \cdot n}{r^3}\, dS = \iint_{\underset{S_2}{\overset{\shortparallel}{\partial B_\varepsilon(\vec{0})}}} \frac{\vec{r} \cdot n}{r^3}\, dS$$

But $\vec{n} = \vec{r}/r$ on $B_\varepsilon(\vec{0})$, so

$$\iint_{\partial B_\varepsilon(\vec{0})} \frac{\vec{r} \cdot n}{r^3}\, dS = \iint_{\partial B_\varepsilon(\vec{0})} \frac{1}{r^2}\, dS = \frac{4\pi\varepsilon^2}{\varepsilon^2} = 4\pi \qquad \blacksquare$$

Physical interpretation: total flux across surface is Q if charge in, 0 out

HW: pg 573: #1, #7, #12, #15, #18, #19, #23

Suggested: #2, #5, #11, #14, #16, #17, #22

1 GENERALIZED STOKES: NOTES ON DIFF FORMS

Chapter 15: C+S: Calc in Vector-Spaces

① Algebra of diff forms

Open $U \subset \mathbb{R}^n$

diff 0-form is a twice cont diff fn $f: U \to \mathbb{R}$

diff 1-form $\sum_{i=1}^{n} f_i \, dx_i$ $\quad f_i$ 0-form

diff k-form = terms like $f \, dx_{i_1} \cdots dx_{i_k}$

addition: $f \, dx_{i_1} \cdots dx_{i_k} + g \, dx_{i_1} \cdots dx_{i_k} = (f+g) \, dx_{i_1} \cdots dx_{i_k}$

mult $g \cdot (f \, dx_{i_1} \cdots dx_{i_k}) = (gf) \, dx_{i_1} \cdots dx_{i_k}$

Rule: $dx_i dx_j = - dx_j dx_i \implies dx_i dx_i = 0$ $\quad$ or 1-form ω $\quad \omega^2 = 0$ $\quad$ Think volume

Ex: Prod of 1-forms

$$\left(\sum f_i dx_i\right)\left(\sum g_j dx_j\right) = \sum_{i,j} f_i g_j \, dx_i dx_j$$
$$= \sum_{1 \le i < j \le n} (f_i g_j - f_j g_i) \, dx_i dx_j$$

② Basic Properties of Sum + Product

ω_1, ω_2 p-forms, η_1, η_2 q-forms, γ r-form

① $(\omega_1 + \omega_2)\eta_1 = \omega_1 \eta_1 + \omega_2 \eta_1$

② $\omega_1(\eta_1 + \eta_2) = \omega_1 \eta_1 + \omega_1 \eta_2$

③ $\omega_1(\eta_1 \gamma) = (\omega_1 \eta_1)\gamma$

④ $\omega_1 \eta_1 = (-1)^{pq} \eta_1 \omega_1$

⑤ ω an n-form on $U \subset \mathbb{R}^n \implies \omega = f \, dx_1 \cdots dx_n$
$\quad \omega$ a p-form with $p > n$ on $\mathbb{R}^n \implies \omega = 0$

⑥ $A = (a_{ij})$ $n \times n$, 1-forms $V_j = \sum_{i=1}^{n} a_{ij} dx_i$
$\quad$ Then $V_1 \cdots V_n = (\operatorname{Det} A) \, dx_1 \cdots dx_n \leftarrow$ follows prop det

GENERALIZED STOKES + DIFF FORMS

③ The Exterior Diff

All fns at least C^2

f 0-form, exterior derivative takes f to 1 form

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

Ex: $f(x_1, x_2, x_3) = x_1^2 + x_2 x_3 \Rightarrow df = 2x_1 dx_1 + x_3 dx_2 + x_2 dx_3$

Ex: $x_i : \mathbb{R}^n \to \mathbb{R}$ by $x_i(y_1, \dots, y_n) = y_i$

so $d(x_i) = \sum_j \frac{\partial x_i}{\partial x_j} dx_j = dx_i$

Extension: ω p-form $\sum_I f_I \, dx_{i_1} \cdots dx_{i_p}$

Then $d\omega = \sum_I (df_I) \, dx_{i_1} \cdots dx_{i_p}$

$$= \sum_I \sum_{j=1}^{n} \frac{\partial f_I}{\partial x_j} dx_j \, dx_{i_1} \cdots dx_{i_p}$$

Examples

$\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$: grad $\vec{F} = \nabla \vec{F} = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$

$f : \mathbb{R}^n \to \mathbb{R}$ div $\vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x_1} + \dots + \frac{\partial F_n}{\partial x_n}$

$$\text{curl} \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right)$$

$\mathcal{U}$ open $\subset \mathbb{R}^n$, $\Omega^p(\mathcal{U})$ all p-forms on $\mathcal{U}$

Isom $\Omega^{n-k}(\mathcal{U}) \approx \Omega^k(\mathcal{U})$

↳ not canonical: Ex: 1-form and $n-1$ form

$$\omega = \sum F_i \, dx_i \longrightarrow \sum (-1)^{i+1} F_i \, dx_1 \cdots dx_{i-1} \, dx_{i+1} \cdots dx_n = \tilde{\omega}$$

why? $d\tilde{\omega} = \left(\sum \frac{\partial F_i}{\partial x_i}\right) dx_1 \cdots dx_n = (\text{div} \vec{F}) dx_1 \cdots dx_n$!

Ex: 0-form f: $df = \sum \frac{\partial f}{\partial x_i} dx_i \Leftrightarrow \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$

Ex: 1 form $\omega = F_1 dx_1 + F_2 dx_2 + F_3 dx_3$

$$d\omega = \frac{\partial F_1}{\partial x_2} dx_2 dx_1 + \frac{\partial F_1}{\partial x_3} dx_3 dx_2 + \frac{\partial F_2}{\partial x_1} dx_1 dx_2 + \frac{\partial F_2}{\partial x_3} dx_2 dx_3 + \cdots$$

$$= \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right) dx_2 dx_3 + \left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}\right) dx_3 dx_1 + \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dx_1 dx_2$$

Generalized Stokes - Notes on Diff Forms

(4) Basic Props of Exterior Diff

① $d(\omega_1+\omega_2)=d\omega_1+d\omega_2$

② ω p-form, η q-form: $d(\omega\eta)=(d\omega)\eta+(-1)^p\omega\, d\eta$
↳ do first two 0-forms, linearity

③ for any p-form ω: $d(d\omega)=0$
↳ enough to do for 0-form f

$$d(df)=d\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i}dx_i\right)=\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_j\partial x_i}dx_j dx_i$$

$$=\sum_{1\le i<j\le n}\left(\frac{\partial^2 f}{\partial x_i\partial x_j}-\frac{\partial^2 f}{\partial x_j\partial x_i}\right)dx_i dx_j=0 \quad \text{as } f \text{ is } C^2$$

(5) Action of differentiable fns on forms

open $U\subset\mathbb{R}^n$, open $V\subset\mathbb{R}^m$, C^2 $\varphi: U\to V$

$\varphi(\vec{u})=(\varphi_1(\vec{u}),\dots,\varphi_m(\vec{u}))$

Want: assoc to φ map φ^* st

(1) $d(\varphi^*\omega)=\varphi^*(d\omega)$

(2) $\varphi^*(\omega+\omega')=\varphi^*(\omega)+\varphi^*(\omega')$ $\quad\omega,\omega'$ p-forms

(3) $\varphi^*(\omega\eta)=\varphi^*(\omega)\,\varphi^*(\eta)$

Defn: f 0-form: $\varphi^*(f)=f\circ\varphi$: note f 0-form on V

$dx_1,\dots,dx_n$ and $dy_1,\dots,dy_m$

special case $n=m$ $\varphi(\vec{u})=\vec{u}$

$$\varphi^*(dy_i)=\sum_{j=1}^n\frac{\partial\varphi_i}{\partial x_j}dx_j$$

$$\varphi^*\left(\sum_I f_I\, dy_{i_1}\cdots dy_{i_p}\right)=\sum_I\varphi^*(f_I)\,\varphi^*(dy_{i_1})\cdots\varphi^*(dy_{i_p})$$

$$\varphi^*:\Omega^p_m(V)\to\Omega^p_n(U)$$

GENERALIZED STOKES: NOTES ON DIFF FORMS

Examples

① $\varphi : \mathbb{R} \to \mathbb{R}$, $\omega = f\,dy$ is a 1-form

$$\varphi^* \omega = \varphi^*(f) \cdot \varphi^*(dy) = (f \circ \varphi) \cdot \frac{d\varphi}{dx} dx$$ chain rule

② $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$

$$\varphi(x_1, x_2) = (Ax_1 + Bx_2,\ Cx_1 + Dx_2)$$

$$\varphi^*(dx_1) = A\,dx_1 + B\,dx_2$$

$$\varphi^*(dx_2) = C\,dx_1 + D\,dx_2$$

$$\varphi^*(dx_1)\,\varphi^*(dx_2) = (AD - BC)\,dx_1\,dx_2$$ change of var!

⑥ Integration of forms

Standard p-cube I^p in $\mathbb{R}^p$: $\{(x_1, \ldots, x_p) : 0 \le x_i \le 1\}$

singular p-cube on $B \subset \mathbb{R}^n$ is diff map $T : I^p \to B$

↳ ex: parametrized curve $\alpha : I^1 \to \mathbb{R}^n$

" surface $T : I^2 \to \mathbb{R}^n$

open $U \subset \mathbb{R}^n$, ω p-form, $T : I^p \to U$

$$\int_T \omega \overset{\text{def}}{=} \int_{I^p} T^* \omega$$

Ex: $\omega = f_1\,dx_1 + \cdots + f_n\,dx_n$, $c : I^1 \to \mathbb{R}^n$, $c(t) = (c_1(t), \ldots, c_n(t))$

$$c^*\omega = (f_1 \circ c)\frac{\partial c_1}{\partial t} dt + \cdots + (f_n \circ c)\frac{\partial c_n}{\partial t} dt = F(c(t)) \cdot c'(t)\,dt$$

so $\int_c \omega = \int_{I^1} c^*\omega = \int_0^1 f(c(t)) \cdot c'(t)\,dt$!

Stokes: $\int_{\partial c} \omega = \int_c d\omega$: Goal is to have "full" integrals