SEQUENCES AND SERIES

GOAL: Many complex phenomena can be well-approx with a series expansion

EXAMPLES:
- Instantaneous interest
- Xeno's (Zeno?) Paradox
- Newton's Method
- Riemann Sums
  - ↳ many limits of sums have exact, closed form answer
    - ↳ proofs frequently follow by Induction

Mathematical Induction:

Given some statement $P(n)$ for $n \in \{0, 1, 2, \ldots\}$, to show $P(n)$ is true for all $n \in \{0, 1, 2, \ldots\}$ it suffices to show the following:

(1) Basis Step: $P(0)$ is true

(2) Inductive Step: Assuming $P(n)$ true Then $P(n+1)$ is true

Proof: $P(0)$ true

$$\frac{P(0) \rightarrow P(1) \text{ true}}{P(1) \text{ true}}$$

and so on and

$$\frac{P(1) \rightarrow P(2) \text{ true}}{P(2) \text{ true}}$$

so on. ▨

Geometric Series

$$S_n = 1 + r + r^2 + \cdots + r^n$$

$$\underline{rS_n =} \quad \underline{\quad r + r^2 + \cdots + r^n + r^{n+1}}$$

$$(1-r)S_n = 1 - r^{n+1}$$

$$\Rightarrow S_n = \frac{1-r^{n+1}}{1-r}$$

$\hookrightarrow$ if $|r|<1$ Then $\lim\limits_{n\to\infty} S_n = \dfrac{1}{1-r}$ $\blacksquare$

Alternate proof.

<u>Basketball Game:</u> A and B alternate with A getting a basket with prob p, B with prob q. First basket wins. Set $r = (1-p)(1-q)$, let $X = \text{Prob}(A \text{ wins})$, assume $0<r<1$.

$$\text{Prob}(A\text{ wins}) = X = p + rp + r^2p + r^3p + \cdots$$
$$X = p(1 + r + r^2 + \cdots)$$

$$\text{Prob}(A\text{ wins}) = X = p + rX \Longrightarrow (1-r)X = p \text{ so } X = \frac{p}{1-r}$$

$$\text{Thus } X = p(1 + r + r^2 + \cdots) = \frac{p}{1-r}$$

$$\text{or } 1 + r + r^2 + \cdots = \frac{1}{1-r} \quad \blacksquare$$

$\hookrightarrow$ Note: Expanded calculation:

$$\text{Prob}(A\text{ wins}) = X = \underset{\substack{\uparrow \\ \text{wins on first shot}}}{p} + \underbrace{(1-p)(1-q)p}_{\substack{\text{wins on second shot, so sequence} \\ \text{is miss, miss, hit}}} + \underbrace{(1-p)(1-q)(1-p)(1-q)p}_{\substack{\text{wins on third shot, so sequence of shots is} \\ \text{miss, miss, miss, miss, hit}}} + \cdots$$

CONVERGENCE TESTS

RATIO TEST: Consider sequence $\{a_n\}$ of positive terms. The series $\sum_{n=0}^{N} a_n$ converges as $N \to \infty$ if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$; if $r > 1$ then the series diverges

Suppose $r<1$, let $\rho = r + \frac{1-r}{2} < 1$

Note that as $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$ and $\rho < r$, for all

n sufficiently large we have $\frac{a_{k+1}}{a_k} < \rho$

Thus $a_{k+1} < \rho a_k$

$$a_{k+2} < \rho a_{k+1} < \rho^2 a_k$$

$$a_{k+3} < \rho a_{k+2} < \rho^2 a_{k+1} < \rho^3 a_k$$

$$\vdots$$

$$a_{k+l} < \rho^l a_k$$

So
$$\sum_{m=0}^{N} a_m = \sum_{m=0}^{k-1} a_m + \sum_{m=k}^{N} a_m$$

$$= \sum_{m=0}^{k-1} a_m + \sum_{l=0}^{N-k} a_{k+l}$$

$$\leq \sum_{m=0}^{k-1} a_m + a_k \sum_{l=0}^{N-k} \rho^l$$

↳ Using the Comparison Test

Comparison and Squeeze Tests

Comparison Test: Sequence $\{b_n\}_{n=0}^{\infty}$ of positive terms whose sum converges, assume seq $\{a_n\}_{n=0}^{\infty}$ satisfies $|a_n| \leq b_n$. Then seq $\{a_n\}_{n=0}^{\infty}$ has a convergent sum; ie, $\lim_{n\to\infty} \sum_{k=0}^{n} a_k$ exists and is finite

Example: Let $b_n = 1/2^n$ and $a_n = (-1)^n/2010 \cdot n!$
Then $\sum_{n=0}^{\infty} a_n$ exists (it actually equals $\frac{1}{2010e}$)

↳ Note: Can weaken comparison tests: all that matters is that have $|a_n| \leq b_n$ for all $n \geq n_0$

Squeeze Thm: Assume $a_n \leq b_n \leq c_n$ and $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} c_n$ exist and are finite. Then $\sum_{n=0}^{\infty} b_n$ exists and is finite.

ROOT TEST

ROOT TEST: LET $\{a_n\}_{n=0}^{\infty}$ be a sequence of positive terms. Then the series $\lim_{N\to\infty} \sum_{n=0}^{N} a_n$ converges if $\lim_{n\to\infty} a_n^{1/n}$ converges to a $\rho < 1$ and diverges if $\rho > 1$.

Proof is similar to that of ratio test.

For n large, $a_n^{1/n} < r$ for some $r < 1$; holds for all large n

↳ Thus for n large, $a_n < r^n$ and win by Comparison Test

↳ if $\rho > 1$, similar argument shows diverge

↳ now $a_n^{1/n} > r > 1$ for all n large ∎

NOTE: RATIO AND ROOT TEST PROVIDE NO INFORMATION IF CORRESPONDING LIMIT IS 1

Example: p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \le 1$

↳ ratio: $\lim_{n\to\infty} \frac{1}{(n+1)^p} \Big/ \frac{1}{n^p} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^p = 1$$

↳ root: also get 1

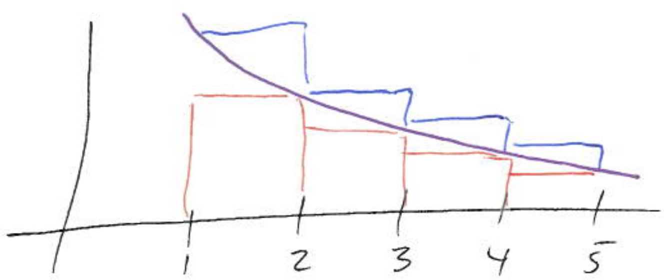
INTEGRAL TEST

INTEGRAL TEST: Sequence $\{a_n\}_{n=0}^{\infty}$ of non-neg terms, assume $f(n) = a_n$ and $f(x)$ is ~~continuous~~ decreasing and integrable.

Then $\sum_{n=0}^{\infty} a_n$ and $\int_0^{\infty} f(x)\,dx$ either both converge or diverge; in many applications start at 1 and not 0.

Note clearly can fail if sequence a_n is not decreasing

Proof: Squeeze Theorem



$$\sum_{n=2}^{\infty} a_n \le \int_1^{\infty} f(x)\,dx \le \sum_{n=1}^{\infty} a_n$$

∎

Example: p-series

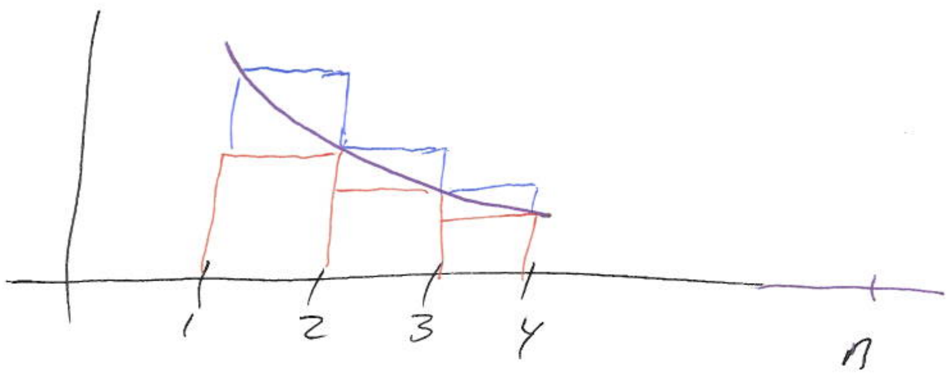
Convergence/divergence of $\sum_{n=1}^{\infty} 1/n^p$ from $\int_1^{\infty} 1/x^p\,dx$

↳ have $\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \frac{x^{1-p}}{1-p}\Big|_1^{\infty} & \text{if } p \neq 1 \\ \ln x \,\Big|_1^{\infty} & \text{if } p = 1 \end{cases}$

↳ Thus converges if $p > 1$, diverges if $p \le 1$

APPROXIMATING SUMS WITH INTEGRALS

Can use the integral test to approximate a sum with an integral, and have some control over the error.



Find $\displaystyle\sum_{k=2}^{n} \frac{1}{k} < \int_1^n \frac{1}{x}\,dx < \sum_{k=1}^{n} \frac{1}{k}$

so $\displaystyle\left(\sum_{k=1}^{n} \frac{1}{k}\right) - 1 < \log n < \sum_{k=1}^{n} \frac{1}{k}$

↳ Thus $\displaystyle\sum_{k=1}^{n} \cancel{\log} \frac{1}{k}$ is within 1 of $\log n$ (and a bit larger). Have $\displaystyle\sum_{k=1}^{n} \frac{1}{k} = \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \cdots$

where $\gamma \approx .5772156$ is the Euler-Mascheroni constant.

<u>Other examples</u>: Stirling's formula for $n!$, very important in probability

<u>Generalization</u>: Better summation formulas (Simpson's rule, Euler-Maclaurin) that (over/under) estimate less do better.

ALTERNATING SERIES

We say a series is absolutely convergent if $\sum_{n=0}^{\infty} |a_n|$ exists and is finite. If $\sum_{n=0}^{\infty} a_n$ exists and is finite but $\sum_{n=0}^{\infty} |a_n|$ diverges (ie, is infinite) we say the sum is conditionally convergent.

↳ Example: $a_n = (-1)^n/n$ is only conditionally convergent

↳ Clear $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$

↳ $$\sum_{n=1}^{\infty} a_n = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$$

$$= -1 + \frac{1}{2\cdot 3} + \frac{1}{4\cdot 5} + \frac{1}{6\cdot 7}$$

$$= -1 + \underbrace{\sum_{n=1}^{\infty} \frac{1}{2n(2n+1)}}_{\text{dominated by } \sum_{n=1}^{\infty} \frac{1}{4n^2} < \infty}$$

note $-1, -1+\frac{1}{2}-\frac{1}{3}, \ldots < \sum_{n=1}^{\infty} a_n < -1+\frac{1}{2}, -1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}$

THM: If $\sum_{n=1}^{\infty} a_n$ is a alternating sum with $|a_n| \to 0$ then the sum converges.

Rearrangement Thm: If $\sum_{n=1}^{\infty} a_n$ converges conditionally but not absolutely, can reorder the terms so that the sum converges to any number you wish!