Notes on sequences and series

Sequence is a collection of terms

$$\{a_n\}_{n=0}^{\infty} = \{a_0, a_1, a_2, a_3 \dots\}$$

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \dots\}$$

Examples: $a_n = 1/n^2 \quad n = 1, 2, 3 \dots$

$$\{a_n\}_{n=1}^{\infty} = \{1, 1/4, 1/9, 1/16, 1/25, \dots\}$$

$b_n = 1/n \quad n = 1, 2, 3, 4, \dots$ (Harmonic sequence)

$$\{b_n\}_{n=1}^{\infty} = \{1, 1/2, 1/3, 1/4, \dots\}$$

Fibonacci numbers:

$a_0 = 1, \quad a_1 = 1, \quad a_{n+1} = a_n + a_{n-1}$

Seq is $\{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$

$\{a_n\}_{n=1}^{\infty}$ with $a_n = n^2$

↳ would be $\{0, 1, 4, 9, 16, \dots\}$

Convergence of a sequence

Say a seq $\{a_n\}_{n=1}^{\infty}$ converges to L

if $\lim\limits_{n\to\infty} a_n = L$.

__Ex__: If $a_n = 1/n$ then the seq converges to 0

since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$.

__Ex__: If $a_n = 2^n$ then the limit does not exist and thus the sequence diverges:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^n = +\infty$$

↳ limit does not exist.

__Ex__: Let $a_n = \frac{3n^2-2n+4}{4n^2+8n+2}$. Does the seq converge?

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2\left(3-\frac{2}{n}+\frac{4}{n^2}\right)}{n^2\left(4+\frac{8}{n}+\frac{2}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{3-\frac{2}{n}+\frac{4}{n^2}}{4+\frac{8}{n}+\frac{2}{n^2}}$$

as we have lim(numerator) = 3 and lim(denom) = 4, use quotient rule for limits:

Quotient rule for limits

$$\lim_{n\to\infty} \frac{b_n}{c_n} = \frac{\lim_{n\to\infty} b_n}{\lim_{n\to\infty} c_n}$$

provided both limits $\lim_{n\to\infty} b_n$ and $\lim_{n\to\infty} c_n$ exist and are finite, and $\lim_{n\to\infty} c_n$ is not zero

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For our problem, just get $3/4$

Alternatively, use L'Hopital

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{6n-2}{8n+8}$$

$$= \lim_{n\to\infty} \frac{6}{8} = \frac{6}{8} = \frac{3}{4}$$

Note: can use L'Hopital so long as you have $0/0$ or ∞/∞; once you no longer have this cannot use L'Hopital.

Ex: $a_n = \dfrac{n^2 + 2n^{3/2} - \log n}{n^3 + 1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n + 3n^{1/2} - \frac{1}{n}}{3n^2}$$

$$= \lim_{n\to\infty} \frac{2 + \frac{3}{2}n^{-1/2} + \frac{1}{n^2}}{6n}$$

$$= 0$$

Stop: No more L'Hopital as $0/\infty$

Note $0/\infty$ is okay: that's just 0

Defn of Series

A series is the sum of terms in a sequence. A partial sum is the sum of the first few terms:

n^{th} partial sum $S_n = \sum_{k=0}^{n} a_k$

Series is $\lim_{n\to\infty} S_n = \sum_{k=0}^{\infty} a_k$

Ex: If $a_n = (1/2)^n \quad n = 0, 1, 2, \dots$

Then
$$\left.\begin{aligned} S_0 &= 1 \\ S_1 &= 1 + \tfrac{1}{2} \\ S_2 &= 1 + \tfrac{1}{2} + \tfrac{1}{4} \\ S_3 &= 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} \end{aligned}\right\} \text{Partial Sums}$$
$$\vdots$$
$$\left. S = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots \right\} \text{Series}$$

When does a series converge?

A neccessary condition for $\sum_{n=0}^{\infty} a_n$ to converge is if each $a_n \to 0$ (ie, $\lim_{n\to\infty} a_n = 0$, ie, the sequence converges to 0).

Note: If $\lim_{n\to\infty} a_n \neq 0$, series diverges

If $\lim_{n\to\infty} a_n = 0$, series may or may not converge

Ex: Consider the following sequence:

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ 1, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{3}, & \frac{1}{3}, & \frac{1}{3}, & \frac{1}{4}, \end{matrix} \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \underbrace{\frac{1}{5}, \dots, \frac{1}{5}}_{5 \text{ times}}, \dots$$

Clearly $\lim_{n\to\infty} a_n = 0$

Note series diverges: $S_1 = 1$, $S_3 = 2$, $S_6 = 3$, $S_{10} = 4$, $S_{15} = 5$, $S_{21} = 6$, $S_{28} = 7, \dots$

Geometric Series

Let $|r| < 1$, let $a_n = r^n$

Then $\displaystyle\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$

More generally, $\displaystyle\sum_{n=0}^{\infty} c\, r^n = \frac{c}{1-r}$

Ex: $r = \frac{1}{2}$: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \dfrac{1}{1-\frac{1}{2}} = 2$

$r = 2$: $1 + 2 + 4 + 8 + \cdots$ diverges

↳ geometric series formula is for $|r| < 1$

Cannot use here as $r > 1$

Proof: $S_n = 1 + r + r^2 + \cdots + r^{n-1} + r^n$

$\underline{\quad r S_n = \quad r + r^2 + \cdots + r^n + r^{n+1}}$

$S_n - r S_n = 1 - r^{n+1}$

$(1-r) S_n = 1 - r^{n+1}$

$S_n = \dfrac{1}{1-r} - \dfrac{r^{n+1}}{1-r}$

$\displaystyle\lim_{n\to\infty} S_n = \frac{1}{1-r} - \frac{1}{1-r}\lim_{n\to\infty} r^{n+1} = \frac{1}{1-r}$

↳ tends to 0 as $|r| < 1$

Ex! $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n - 3^n}$ Converges

Reason: Claim n big:

$$4^n > 2 \cdot 3^n$$

$\hookrightarrow$ take logs: $n \log 4$ vs $n \log 3 + \log 2$

or $n \log \frac{4}{3}$ vs $\log 2$

or n vs $\dfrac{\log 2}{\log \frac{4}{3}}$

So if $n > \dfrac{\log 2}{\log \frac{4}{3}}$, win!

$$4^n > 2 \cdot 3^n \Rightarrow 4^n - 3^n > 2 \cdot 3^n - 3^n$$

or $4^n - 3^n > 3^n$

or $\dfrac{1}{3^n} > \dfrac{1}{4^n - 3^n}$ comparison test

Ex: Try $a_n = r^n$ with Ratio Test

Ex: Try $a_n = \frac{1}{n}$ or $\frac{1}{n^2}$ with Ratio Test

↳ sadly get $\rho = 1$ both times

Ex: Try $a_n = \frac{1}{n!}$: get $\rho = 0$

Proof: $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$= \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \cdot \frac{n!}{n!}$ or use $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$

$= \lim_{n\to\infty} \frac{n!}{(n+1)!}$

$3! = 3 \cdot 2 \cdot 1$
$4! = 4 \cdot 3!$
$0! = 1$

$= \lim_{n\to\infty} \frac{1}{n+1}$

$= 0$