MATH 105: EXTRA CREDIT: SLOPES OF PERPENDICULAR LINES

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Prove that the product of the slopes of two perpendicular lines in the plane that are not parallel to the coordinate axes is -1. What is the generalization of this to lines in three-dimensional space? What is the analogue of the product of the slopes of the line equaling -1?

Let's first consider the case when the two lines intersect at the origin, $(0, 0)$. We need to find a second point on each line. Without loss of generality, we might as well take our second points to be one unit from the origin. Thus we can write the points as $(\cos\theta, \sin\theta)$ for the first line and $(\cos\phi, \sin\phi)$ for the second, where θ and ϕ denote the angle of our line with the positive x-axis. As the two lines are perpendicular, $\phi = \theta + \pi/2$.

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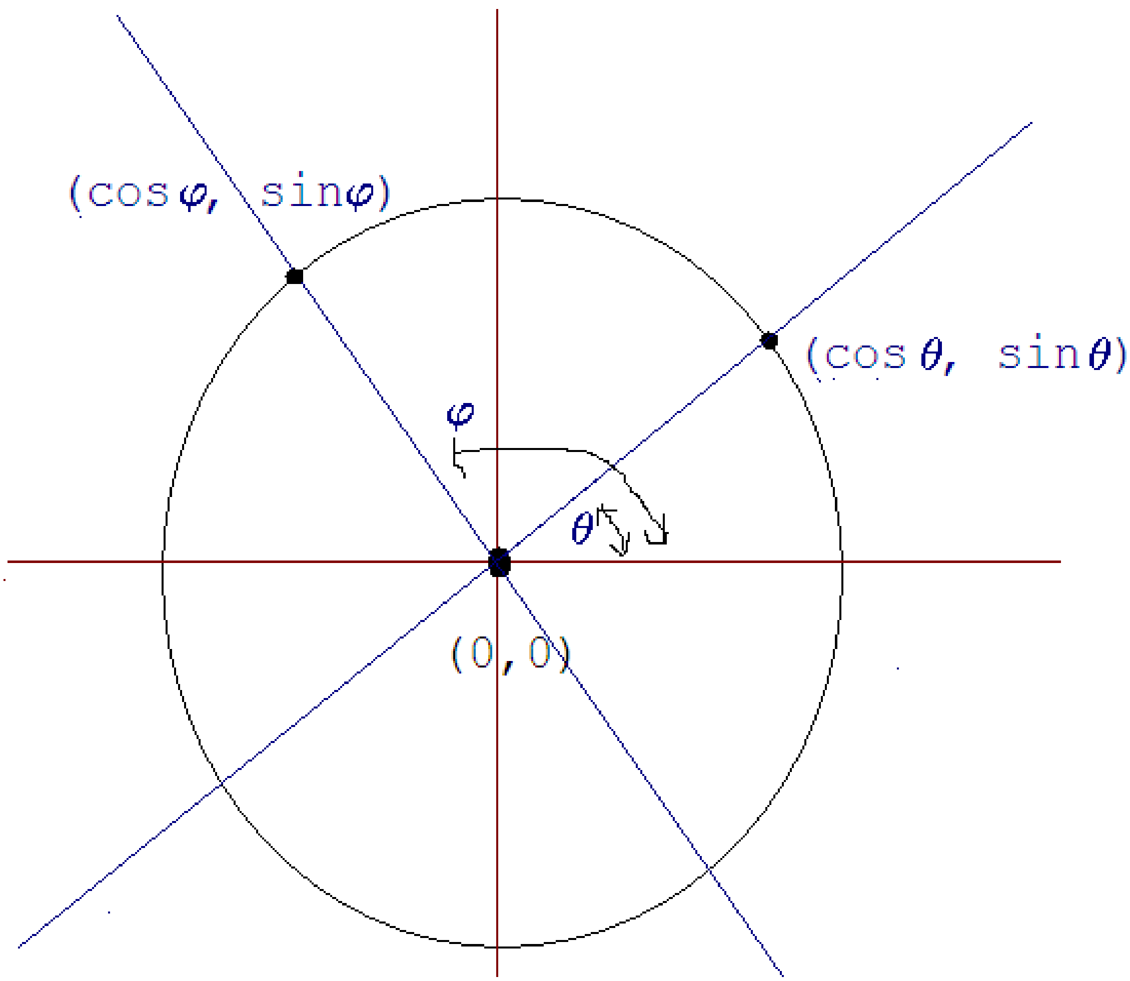

Thus the first line goes through $(0,0)$ and $(\cos\theta, \sin\theta)$. The slope is therefore $\sin\theta/\cos\theta = \tan\theta$. The second line goes through $(0,0)$ and $(\cos(\theta+\pi/2), \sin(\theta+\pi/2))$, and the slope is therefore

$$\frac{\sin(\theta+\pi/2)}{\cos(\theta+\pi/2)} = \frac{\sin\theta\cos\frac{\pi}{2}+\cos\theta\sin\frac{\pi}{2}}{\cos\theta\cos\frac{\pi}{2}-\sin\theta\sin\frac{\pi}{2}} = \frac{\cos\theta}{-\sin\theta} = -\frac{1}{\tan\theta}.$$

Thus the product of the slopes is

$$\tan\theta\cdot\frac{-1}{\tan\theta} = -1$$

as claimed. We leave it to the reader to handle the more general case (i.e., when the two lines do not intersect at the origin – you should be able to argue that moving the intersection to the origin does not change the slopes).

How is this related to the three-dimensional generalization? Two lines are perpendicular if the angle between them is 90 degrees or $\pi/2$ radians. This means the dot product of their directions vanish. Again, let's assume the two lines intersect at the origin, which is now $(0,0,0)$ and that the first line is in the direction $\overrightarrow{v}$ and the second in the direction $\overrightarrow{w}$. Then $\overrightarrow{v}\cdot\overrightarrow{w} = 0$, or $v_1w_1 + v_2w_2 + v_3w_3 = 0$.

How is this related to the product of the slope of two perpendicular lines in the plane being -1? Recall that we may regard a line with slope m in the plane to have direction vector $(1,m)$. We saw this by arguing as follows: for every step of 1 unit we take in the x-direction, we move m unit steps in the y-direction. Thus, in the plane our two lines have direction vectors $(1,m)$ and $(1,n)$, say. As they are perpendicular, the dot product of the two direction vectors vanish, and hence $1\cdot 1 + m\cdot n = 0$, or $mn = -1$, which means the product of the slopes is -1!

I like this problem for several reasons. The first is that it highlights the different ways we have of looking at lines. The second is that it suggests that we must be *very* careful whenever we have certain combinations, such as $\infty - \infty$, $0\cdot\infty$ or ∞/∞. For example, what is $0\cdot\infty$? It depends on the problem. If we look at $\lim_{x\to 0} x\cdot\frac{1}{x}$ we get 1, while $\lim_{x\to 0} x\frac{1}{x^3}$ is ∞ and $\lim_{x\to 0} x\frac{1}{\sqrt{x}}$ is 0. Note that the limit of a product is the product of the limits only when both limits are finite. We may interpret our slope problem as saying that, in this case, $0\cdot\infty$ is -1. The reason is that the product of the slopes of two perpendicular lines not aligned with the coordinate axes is 0; in the limit as the two lines approach the coordinate axes, since the product is always -1 it is not unreasonable to define it to be -1 in the limit as well; however, the slope of the x-axis is 0 and the slope of the y-axis is infinity.