MATH 105: OPTIMIZATION LECTURE

LAGRANGE MULTIPLIERS AND APPLICATIONS

• CALC I REVIEW

To find optima (max/min) check and see which largest/smallest
- ↳ critical points (where $f'(x) = 0$)
- boundary points (usually endpoints of interval)
  - ↳ usually only two, not bad
    - ↳ critical points b/c of interpretation f'
      - ↳ f' pos then locally
      - f' neg then locally

• GENERALIZATION TO SEVERAL VARS

Have some region in $\mathbb{R}^n$, say  or 

To find optima (max/min) check which largest/smallest
- ↳ interior critical points (where $\nabla f = \vec{0}$)
- boundary points
  - ↳ much harder as infinitely many!

Three trouble spots on the spherical Earth, located at

$P_1 = \left(\frac{5}{13}, \frac{12}{13}, 0\right)$ $P_2 = \left(\frac{12}{13}, \frac{5}{13}, 0\right)$ and

$P_3 = \left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$

Method 1: Use "burrowing" distance squared

$$D_1(x,y,z) = 13 \sum_{i=1}^{3} \|(x,y,z) - P_i\|^2$$

↳ factor 13 clears denoms, harmless

↳ working with squares a real issue

$$\nabla D_1 = (78x - 40,\ 78y - 42,\ 78z - 24)$$

Constraint: $g(x,y,z) = x^2 + y^2 + z^2 = 1$

$\nabla g = (2x, 2y, 2z)$

so $\nabla D_1 = \lambda \nabla g$

↳ yields $\lambda = 39 - \sqrt{985}$

$x = 4\sqrt{\frac{5}{197}} \approx .637$

$y = \frac{21}{\sqrt{985}} \approx .669$

$z = \frac{12}{\sqrt{985}} \approx .382$

Yields $D_1 \approx 15.23$

$\lambda = 39 + \sqrt{985}$

$x = -4\sqrt{\frac{5}{197}}$

$y = -\frac{21}{\sqrt{985}}$

$z = -\frac{12}{\sqrt{985}}$

Yields $D_1 \approx 140.77$

Note $\sqrt{\frac{D_1}{13}} \approx 1.08$

Method 2: Don't use distance squared, use distance burrowing through the Earth.

Quickly leads to an algebraic nightmare! Square-roots all throughout the denominator.

Numerically approximate answer:

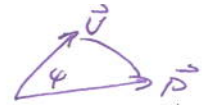
Using $D_2(x,y,z) = \sum_{i=1}^{3} ||(x,y,z) - P(i)||$

↳ $x \approx .6588$

$y \approx .7232$

$z \approx .2071$

Minimum value is 1.8

Method 3: Use $\vec{P}\cdot\vec{U} = \cos\psi$ on sphere 

And note the arclength is $\psi = \arccos(\vec{P}\cdot\vec{U})$

As arccos is a pain to work with that we will minimize

$$D_3(x,y,z) = 2\left[(1-\cos\psi_1) + (1-\cos\psi_2) + (1-\cos\psi_3)\right]$$

For small angles, $2(1-\cos\psi) \approx \psi^2 \approx$ arc length squared!

$$D_3(x,y,z) = 2\sum_{i=1}^{3}\left[1-(x,y,z)\cdot P(i)\right] * 13$$

↳ 13 harmless - clears denominators

$$= 78 - 40x - 42y - 24z$$

$$\nabla D_3 = (-40,-42,-24) = \lambda(2x,2y,2z) = \lambda\nabla g$$

Thus $x = -\frac{20}{\lambda}$, $y = -\frac{21}{\lambda}$, $z = -\frac{12}{\lambda}$

and $x^2+y^2+z^2=1 \rightarrow \frac{20^2+21^2+12^2}{\lambda^2} = 1 \rightarrow \lambda = \pm\sqrt{985}$

Clearly minimum is when $\lambda = -\sqrt{985}$

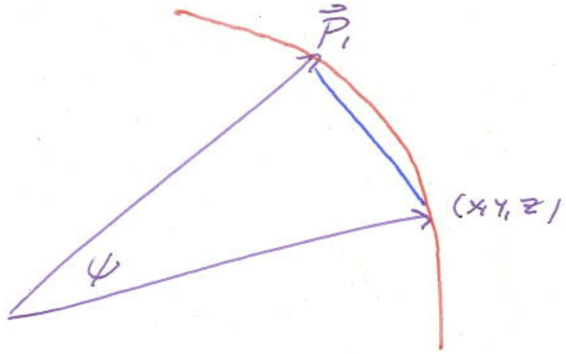
Thus $x = \frac{20}{\sqrt{985}} \approx .637$

$y = \frac{21}{\sqrt{985}} \approx .669$

$z = \frac{12}{\sqrt{985}} \approx .382$

$D_3 = 15.2306$

Why is Method 3 the same as Method 1?



distance thru Earth, by law of cosines, is

$$d^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot \cos\psi$$

Thus $d^2 = 2 - 2\cos\psi$

$$d^2 = 2(1 - \cos\psi)$$

And hence our two functions, D_1 and D_3, are in fact the same!

Method 4 : Cross Product and Sines

We have, since everything is a unit vector,

$$\|P_i \times (x,y,z)\| = \sin\psi \approx \psi \text{ for small } \psi$$

$$D_4(x,y,z) = 13 \sum_{i=1}^{3} \| P_i \times (x,y,z)\|^2$$

$$= \tfrac{1}{2} \left\{ 329x^2 - 264xy + 322y^2 - 72xz - 96yz + 363z^2 \right.$$

↳ In Mathematica, $Norm((x,y,z) \times P(i))^2$ was bad, involving abs values; better to code as $((x,y,z) \times P(i)) \cdot ((x,y,z) \times P(i))$

$$\nabla D_4 = (658x - 264y - 72z, -264x + 644y - 96z, -72x - 96y + 726z)$$

Solve $\nabla D_4 = \lambda \nabla g$

Minimum is $x \approx .659$

$y \approx .689$

$z \approx .301$

Value $D_4 \approx 174.6$

Method 1: Burrowing through the Earth (distance^2)
Method 2: Burrowing through the Earth (distance)
Method 3: Using 1 – cos(angle).
Method 4: Using sin(angle)

	Method 1	Method 2	Method 3	Method 4
x	.637	.659	.637	.659
y	.669	.723	.669	.689
z	.382	.207	.382	.302

Trouble points are:

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p[1] = {5/13,12/13,0};
p[2] = {12/13,5/13,0};
p[3] = {3/13,4/13,12/13};
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