SECTION 2.4: INTRO TO PATHS AND CURVES

<u>Path in $\mathbb{R}^n$</u> is a map $c : [a,b] \to \mathbb{R}^n$. Collection C of points $c(t)$ is a <u>curve</u> with endpoints $c(a)$, $c(b)$.
The path c <u>parametrizes</u> the curve C.

↳ Ex: $n=2$: $c(t) = (x(t), y(t))$

Ex: line: $c(t) = \vec{a} + t\vec{v}$

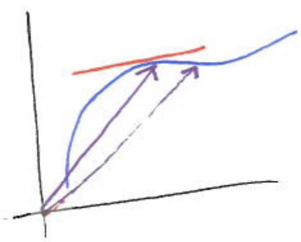
Ex: circle: $c_1(t) = (\cos t, \sin t) \quad t \in [0, 2\pi]$

$c_2(t) = (\cos 2t, \sin 2t) \quad t \in [0, \pi]$

$c_3(t) = (\sin t, \cos t) \quad t \in [0, 2\pi]$

↳ different paths parametrize same curve

<u>Velocity vector</u>: If c is a differentiable path then velocity of c at time t is $c'(t) = \lim_{h \to 0} \frac{c(t+h) - c(t)}{h}$



↳ denote speed by $s(t) = \|c'(t)\|$

↳ tangent line at time t_0 is $l(t) = c(t_0) + c'(t_0)(t - t_0)$

Homework: #1, #7, #15

Suggested: #3, #6, #13, #17

SUPPLEMENT TO SECTION 2.4

Below are some items related to the lecture on Section 2.4. This is just a quick sketch of the additional items from the lecture not in the quick lecture note summary.

Applications:
- Planetary motion (orbits, probes)
- Ballistics (projectiles, cannon balls, baseball)

Example: Parametrizing Circle

- Regard as level set of $f(x,y)$ with value r^2 (here $f(x,y) = x^2 + y^2$).
- Want to solve $x(t)^2 + y(t)^2 = r^2$ for circle
- What do we need to parametrize a circle?
  - ↳ radius r
  - ↳ center (x_0, y_0)
  - ↳ speed and direction: velocity
  - ↳ starting point

Ex: Circle of radius 2, center $(0,0)$, start at $(2,0)$, counterclockwise

↳ Soln: $\vec{c}(t) = (x(t), y(t)) = (2\cos t, 2\sin t)$

↳ check: clear $x(t)^2 + y(t)^2 = 4$

must show get all points as time passes

↳ show given any point there is a t

Supplement to Section 2.4 (cont)

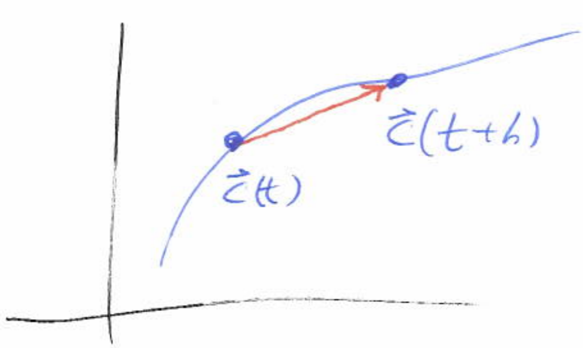
Consider: • $\vec{c}(t) = (2\cos t, 2\sin t)$

$$\vec{d}(t) = (2\cos(-t), 2\sin(-t))$$
$$= (2\cos t, -2\sin t)$$
$$\vec{e}(t) = (2\cos(2\pi t), 2\sin(2\pi t))$$

↳ Physically all trace out same circle
  ↳ $\vec{c}$ and $\vec{e}$ do it clockwise, $\vec{d}$ counter clockwise

↳ Note $\vec{e}$ is faster than $\vec{c}$ and $\vec{d}$: takes 1 unit of time to trace out, as opposed to 2π for $\vec{c}, \vec{d}$
  ↳ leads to studying speed of a parametrization
    ↳ often useful travel unit speed
      ↳ lots of eqs involve speed, if unit and forget not a big deal
      ↳ ex: angle b/w $\vec{v}$ and $\vec{w}$ is: $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$
        bad if forget lengths unless equal!

Tangent Vector



$\vec{c}(t+h) - \vec{c}(t)$ is approx distance traveled in time h, gets better as time shrinks to 0. Thus instantaneous speed is well approx by $\|\vec{c}(t+h) - \vec{c}(t)\| / h$, and speed $= \|\vec{c}\,'(t)\|$

Supplement to Section 2.4 (cont)

Want to reparametrize to unit speed

Consider $\vec{c}(t) = (2\cos t,\ 2\sin t)$

↳ have $\vec{c}\,'(t) = \cancel{4\cos}\ (-2\sin t,\ 2\cos t)$

$$\|\vec{c}\,'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t}$$

$$= \sqrt{4} = 2$$

This traveling at say 2 units/sec. How can we adjust to travel at 1 unit/sec?

Attempt 1: $\vec{h}_1(t) = \frac{1}{2}\vec{c}(t) = (\cos t, \sin t)$

↳ natural to try. Do get $\|\vec{h}_1'(t)\| = 1$, but no longer a circle of radius 2! Cannot change output: must trace same physical shape.

Attempt 2: $\vec{h}_2(t) = \vec{c}(at) = (2\cos(at), 2\sin(at))$

↳ natural: adjusting how rapidly time passes!

$$\vec{h}_2'(t) = (-2a\sin(at),\ 2a\cos(at))$$

$$\|\vec{h}_2'(t)\| = \sqrt{4a^2\sin^2(at) + 4a^2\cos^2(at)} = 2|a|$$

↳ Thus $a = 1/2$ (orientation preserving) or $a = -1/2$ (reversing)

Example: parametrize level set of value 4 to $f(x,y) = x^2+4y^2$

- If had $x^2+y^2=4$ a circle, should be similar

Try $x(t) = a\cos t$ $\quad y(t) = b\sin t$

Get $\vec{c}(t) = (a\cos t, b\sin t)$

↳ $x(t)^2 + 4y(t)^2 = a^2\cos^2 t + 4b^2\sin^2 t$

↳ if $a^2 = 4b^2$ Then have same number of cosines and signs : Pythagoras simplifies

If $a^2 = 4b^2 = 4$ Then works!

Thus soln is $a = \pm 2$, $b = \pm 1$

↳ Ex: take $\vec{c}(t) = (2\cos t, \sin t)$

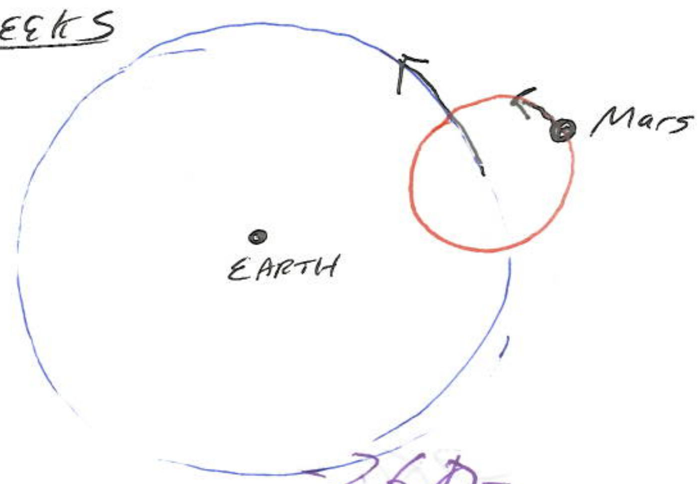
↳ parametrizes ellipse!

↳ Question: Constant speed?

$\vec{c}\,'(t) = (-2\sin t, \cos t)$

$\|\vec{c}\,'(t)\| = \sqrt{4\sin^2 t + \cos^2 t} = \sqrt{1+3\sin^2 t}$: NO!

BEAT THE GREEKS



circles on circles....