HW #6

2.2 #5, #8 ab, #17 2.3 #1ad

5) Compute the limits:

a) $\lim\limits_{(x,y)\to(0,1)} x^3 y = 0$ since $(0)^3(1) = \boxed{0}$

b) $\lim\limits_{x\to 0} \frac{\cos x - 1}{x^2} = \lim\limits_{x\to 0} \frac{-\sin(x)}{2x} = \left(\lim\limits_{x\to 0} \frac{-1}{2}\right)\left(\lim\limits_{x\to 0} \frac{\sin(x)}{x}\right) = \left(-\frac{1}{2}\right)(1) = \boxed{-\frac{1}{2}}$

c) $\lim\limits_{h\to 0} \frac{e^h - 1}{h} = \lim\limits_{h\to 0} \frac{e^h}{1} = \lim\limits_{h\to 0} e^h = \boxed{1}$

8) a) $\lim\limits_{(x,y)\to(0,0)} \frac{(x+y)^2-(x-y)^2}{xy} = \lim\limits_{(x,y)\to(0,0)} \frac{(x^2+2xy+y^2)-(x^2-2xy+y^2)}{xy}$

$= \lim\limits_{(x,y)\to(0,0)} \frac{4xy}{xy} = \lim\limits_{(x,y)\to(0,0)} 4 = \boxed{4}$

b) $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(xy)}{y}$ there are 2 ways to approach this. Either is correct

i) consider when $y=0$. Then $\frac{\sin(xy)}{y}$ is undefined. Since the function is undefined on one approach, the limit does not exist.

ii) consider what happens when we multiply by $1 = \left(\frac{x}{x}\right)$

$\lim\limits_{(x,y)\to(0,0)} \frac{\sin(xy)}{y}\left(\frac{x}{x}\right) = \lim\limits_{(x,y)\to(0,0)} \frac{x\sin(xy)}{xy}$

$= \left(\lim\limits_{(x,y)\to(0,0)} x\right)\left(\lim\limits_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy}\right) = (0)(1) = 0$

17) a) Can $\frac{\sin(x+y)}{x+y}$ be made continuous by suitably defining it at (0,0)?

yes. Consider $\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x+y}$. let $h = x+y$

$\lim_{h\to 0} \frac{\sin(h)}{h} = 1$ So by defining $\frac{\sin(x+y)}{x+y}$ as 1 at (0,0) we can make the function continuous.

b) Can $\frac{xy}{x^2+y^2}$ be made continuous by suitably defining it at (0,0)?

no. Consider the following approaches: $x=y$ and $x=-y$

when $x=y$: $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$

when $x=-y$: $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{-x^2}{2x^2} = -\frac{1}{2}$

since $\frac{1}{2} \neq -\frac{1}{2}$, the function cannot be made continuous at (0,0)

c) Prove that $f: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto ye^x + \sin x + (xy)^4$ is continuous.

We know that separately y, e^x, $\sin x$, x are all continuous.

Since the product of continuous functions is continuous, ye^x, $(xy)^4$ are continuous.

Since the sum of continuous functions is continuous, $ye^x + \sin x + (xy)^4$ is continuous.

1) find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for:

a) $f(x,y) = xy$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} xy = y$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} xy = x$$

d) $f(x,y) = (x^2+y^2)\log(x^2+y^2)$

note: if no base is specified, $\log(u) = \ln(u)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2)\log(x^2+y^2)$$

let $u = x^2+y^2$ $\quad v = \log(x^2+y^2)$

$\frac{\partial u}{\partial x} = 2x$ $\quad \frac{\partial v}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x$

$$\frac{\partial f}{\partial x} = u\,\partial v + v\,\partial u = \cancel{(x^2+y^2)}\left(\frac{1}{\cancel{x^2+y^2}} \cdot 2x\right) + \log(x^2+y^2) \cdot 2x$$

$$= 2x + \log(x^2+y^2) \cdot 2x$$

$$= 2x\,(1 + \log(x^2+y^2))$$

similarly, because of symmetry

$$\frac{\partial f}{\partial y} = 2y\,(1 + \log(x^2+y^2))$$