Solutions to HW # 10

4) $f(u,v) = \frac{u^2+v^2}{u^2-v^2}$ $\quad u(x,y) = e^{-x-y}$ $\quad v(x,y) = e^{xy}$

Find $\frac{\partial h}{\partial x}$ where $h(x,y) = f(u(x,y), v(x,y))$

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial u} = \frac{(u^2-v^2)(2u) - (u^2+v^2)(2u)}{(u^2-v^2)^2} = \frac{2u(u^2-v^2-u^2-v^2)}{(u^2-v^2)^2} = \frac{-4uv^2}{(u^2-v^2)^2}$$

$$\frac{\partial f}{\partial v} = \frac{(u^2-v^2)(2v) + (u^2+v^2)(2v)}{(u^2-v^2)^2} = \frac{2v(u^2+u^2)}{(u^2-v^2)^2} = \frac{4vu^2}{(u^2-v^2)^2}$$

$$\frac{\partial u}{\partial x} = -e^{-x-y} \qquad \frac{\partial v}{\partial x} = e^{xy}$$

$$\frac{\partial h}{\partial x} = \frac{-4uv^2}{(u^2-v^2)^2}(-e^{-x-y}) + \frac{4vu^2}{(u^2-v^2)^2}(e^{xy}) = \frac{4uv}{(u^2-v^2)^2}\left(ve^{-x-y} + u(e^{xy})\right)$$

$$= \frac{4e^{xy-x-y}}{(e^{-2(x+y)} - e^{2xy})}\left(2e^{xy-x-y}\right) = \frac{6e^{xy-x-y}}{(e^{-2(x+y)} - e^{2xy})}$$

5a) $f(x,y) = xy$ $\quad c(t) = (e^t, \cos t)$

$$\frac{\partial h}{\partial t} = \nabla f(c(t)) \cdot c'(t) = (\cos t, e^t) \cdot (e^t, -\sin t)$$
$$= \cos t\, e^t - \sin t\, e^t$$

13) $x = \cos t \quad y = \sin t \qquad T = x^2 e^y - xy^3$

a) $\frac{dT}{dt} = \nabla T(c(t)) \bullet c'(t) = (2xe^y - y^3,\ x^2e^y - 3xy^2)$
$\bullet (-\sin t, \cos t) = (2\cos t\, e^{\sin t} - \sin^3 t,\ \cos^2 t\, e^{\sin t} - 3\cos t \sin^2 t)$
$\bullet (-\sin t, \cos t)$
$= \sin^4 t - 2e^{\sin t}\cos t \sin t + e^{\sin t}\cos^3 t - 3\cos^2 t \sin^2 t$

b) $T(t) = \cos^2 t\, e^{\sin t} - \cos t \sin^3 t$

$\frac{dT}{dt} = e^{\sin t}\cos^3 t - 2e^{\sin t}\cos t \sin t - 3\sin^2 t \cos^2 t + \sin^4 t$

20) $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

a) Using the definition of the derivative:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$f(0,0) = 0$

$$\frac{\partial f(0,0)}{\partial x} = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{0}{h^2}}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

Similarly,

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{0}{h^2}}{0} = 0$$

b) $g(t) = (at, bt)$

$$(f \circ g)' = \frac{at(bt)^2}{(at)^2 + (bt)^2} = \frac{ab^2 t}{(a)^2 + (b^2)} \frac{(t^2)}{(t^2)} = \frac{ab^2}{a^2+b^2}(t)$$

$$(f \circ g)'(t) = \frac{ab^2}{a^2+b^2}$$

$$(f \circ g)'(0) = \frac{ab^2}{a^2+b^2}$$

To find $\nabla f(0,0) \cdot g'(0) = 0$, recall from (a) that:

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

so, $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (0,0)$

$g'(t) = (a, b)$

$$\nabla f(0,0) \cdot g'(0) = (0,0) \cdot (a,b) = 0 + 0 = 0$$