HW # 11 Solutions

2. a) $f(x,y) = x + 2xy + -3y^2$ $(x_0, y_0) = (1,2)$
$\vec{v} = \left(\frac{3}{5}, \frac{4}{5}\right)$

First, find the partials at the given pt.

So $\frac{\partial f}{\partial x} = 1 + 2y \Big|_{(1,2)} = 5$

$\cancel{\frac{\partial f}{\partial x}(1,2)} = \frac{\partial f}{\partial y} = 2x - 6y \Big|_{(1,2)} = -10$

To find the directional derivative, we take the dot product of ∇f and $\vec{v}$. Such that

$\cancel{\nabla f(x,y)} = (5, -10) \cdot (3/5, 4/5)$
$= (3 + -8) = -5$

b) To ~~do~~ solve b, we follow similar procedures,

$F(x,y) = \log\sqrt{x^2+y^2} = \log(x^2+y^2)^{1/2}$
(which by log rules) $= \frac{1}{2}\log(x^2+y^2)$

$\frac{\partial f}{\partial x} = \frac{1}{2} \cdot \frac{1}{(x^2+y^2)} \cdot 2x = \frac{x}{(x^2+y^2)} \Big|_{(1,0)} = 1$

$\frac{\partial f}{\partial y} = \frac{y}{(x^2+y^2)} \Big|_{(1,0)} = 0$

So, if $\nabla f(1,0) = (1,0)$ and directional derivative
$\nabla f \cdot \vec{v} = (1,0) \cdot \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$
$= \frac{2}{\sqrt{5}}$

4 a). $x^2 + 2y^2 + 3xz = 10$ at $(1, 2, \frac{1}{3})$

Equation to find the tangent plane:

$$\nabla f(x,y,z) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

So first we find the gradient,

$$\frac{\partial f}{\partial x} = 2x + 3z \Big|_{(1,2,\frac{1}{3})} = 3 \quad \frac{\partial f}{\partial y} = 4y \Big|_{(1,2,\frac{1}{3})} = 8 \quad \frac{\partial f}{\partial z} = 3x \Big|_{(1,2,\frac{1}{3})} = 3$$

So $\nabla f(3,8,3)$ and $(x_0, y_0, z_0) = (1, 2, \frac{1}{3})$

So equation is

$$(3,8,3) \cdot (x-1,\ y-2,\ z-\frac{1}{3}) = 0$$

$$3(x-1) + 8(y-2) + 3(z - \frac{1}{3}) = 0$$

which simplifies to

$$3x + 8y + 3z = 20.$$

6. a) Compute Gradient for

$$f(x,y,z) = \frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}} = (x^2+y^2+z^2)^{-\frac{1}{2}}$$

So you need to use chain rule

$-\frac{1}{2} \cdot \frac{2}{2}$

$$\frac{\partial f}{\partial x} = \frac{-1}{\cancel{2}} (x^2+y^2+z^2)^{-3/2} \cdot \cancel{2}x = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

similarly,

$$\frac{\partial f}{\partial y} = \frac{-y}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial f}{\partial z} = \frac{-z}{(x^2+y^2+z^2)^{3/2}}$$

16. $T(x,y,z) = e^{-x^2-2y^2-3z^2}$

a) find $\nabla f(1,1,1)$ (this is the direction of fastest <u>increase</u> so to find direction of fastest <u>decrease</u> we take $-\nabla f$.

$$\frac{\partial f}{\partial x} = e^{-x^2-2y^2-3z^2}(-2x)\Big|_{(1,1,1)} = -2e^{-6}$$

$$\frac{\partial f}{\partial y} = (-4y)e^{-x^2-2y^2-3z^2}\Big|_{(1,1,1)} = -4e^{-6}$$

$$\frac{\partial f}{\partial z} = (-6z)e^{-x^2-2y-3z}\Big|_{(1,1,1)} = -6e^{-6}$$

$$\nabla f(1,1,1) = (-2e^{-6}, -4e^{-6}, -6e^{-6})$$

$$-\nabla f(1,1,1) = 2e^{-6}, 4e^{+6}, 6e^{-6})$$

b) Find $\|\nabla f(1,1,1)\| \cdot e^8$

$$\|\nabla f(1,1,1)\| = \sqrt{(2e^{-6})^2 + (4e^{-6})^2 + (6e^{-6})^2}$$

$$= \sqrt{4e^{-12} + 16e^{-12} + 36e^{-12}}$$

$$\cdot e^8 \sqrt{56e^{-12}} = 2\sqrt{14}\, e^2$$

c) $2e^2\sqrt{14}\cos\theta \leq e^2\sqrt{14}$

$$\theta = \cos^{-1}\left(\frac{e^2\sqrt{14}}{2e^2\sqrt{14}}\right)$$

$$\theta = 60^\circ$$

18. $\nabla f(x) = g(x)\vec{x}$

here, the function $g(x)$ is a scalar so the direction of $\nabla f(x)$ is the same as that of vector X.

* Note. the normal to a sphere ~~to~~ at the point $\bar{x}$ is also the vector $\bar{x}$ so ~~g~~ ∇f is always in the direction of the normal to the sphere.