Alan Arias

Solutions to HW #27

10.7

13) Find all p for which $\sum_{k=1}^{n} \frac{1}{k^p}$ converges and diverges.

We have already shown that $p=1$ is the harmonic series, which diverges. If $p \leq 0$, then $a_n \geq 1$ for all n, producing a divergent infinite series. All that is left is to show for $p > 1$ and $0 < p < 1$. We use the root test.

$$* \quad \int_1^{\infty} \frac{1}{k^p}\,dk = \lim_{k\to\infty} \left[-\frac{1}{(p-1)\,x^{p-1}} \right]_1^R$$

$$= \lim_{R\to\infty} \frac{1}{p-1} \left(1 - \frac{1}{b^{p-1}} \right)$$

If $p > 1$, then:

$$\int_1^{\infty} \frac{1}{k^p}\,dk = \frac{1}{p-1} < \infty$$

so the integral and the series both converge. But if $0 < p < 1$, then

$$\int_1^{\infty} \frac{1}{k^p}\,dk = \lim_{R\to\infty} \frac{1}{1-p} \left(b^{1-p} - 1 \right) = \infty$$

in which case both integral and the series diverge.

For $p > 1$, the series converges. For $p < 1$, the series diverges.

* From Edwards and Penney, Multivariable Calculus, 6th Ed, p717

Prof Miller

#14) $\sum_{k=0}^{n} \frac{1}{2e^k+k}$ As $n \to \infty$, Converges

Proof: Simplest is comparison test.

Note $2e^k + k \geq e^k$

So $\frac{1}{2e^k+k} \leq \frac{1}{e^k} = \left(\frac{1}{e}\right)^k$

As $\sum_{k=0}^{\infty} \left(\frac{1}{e}\right)^k$ converges (geometric series

with ratio $\frac{1}{e} < 1$), our sum converges

note Ratio test would work too.

#15) $\sum_{k=1}^{\infty} \frac{1}{2k+1}$ As $n \to \infty$, converges

Proof: Simplest is comparison test.

Note $2k+1 \leq 4k$ for $k \geq 1$

Thus $\frac{1}{4k} \leq \frac{1}{2k+1}$

As $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, so too does $\sum_{k=1}^{\infty} \frac{1}{4k}$, and

Thus by the comparison test $\sum_{k=1}^{\infty} \frac{1}{2k+1}$ diverges

Note: root and ratio tests would provide no info,

but the integral test would work.

Take $f(x) = \frac{1}{2x+1}$ and

$F(x) = \int f(x)dx$ is $\frac{1}{2}\log(2x+1)$

Prof Miller

#16) $\displaystyle\sum_{n=2}^{\cancel{\infty}\, n} \frac{1}{\log k}$ as $n \to \infty$, diverges

Proof: Ratio, root, Integral tests won't work or are hard (what function has derivative $\frac{1}{\log x}$?)

Comparison test is easiest:

$$\log k \leq k \quad \text{for } k \geq 2$$

so $$\frac{1}{k} \leq \frac{1}{\log k}$$

As $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k}$ diverges, so too does $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\log k}$

Note: if we could take $k = e^n$, then $\frac{1}{\log k} = \frac{1}{n}$ and this would give the harmonic sum as a subset of our sum, which diverges! Of course, k must be an integer so we can look at the closest integer k to e^n for each integer.

17) Use the comparison test with:

$$\frac{1}{k^2+k-1} < \frac{1}{k^2} \quad \text{for large } k. \ (k \geq 1 \text{ works!})$$

$\sum_{k=0}^{n} \frac{1}{k^2}$ converges, so $\sum_{k=0}^{n} \frac{1}{k^2+k-1}$ converges.

Page 10, 10

18. Use the Ratio Test:

$$\lim_{k\to\infty} \frac{a^{k+1}}{a^k} = \lim_{k\to\infty} \frac{10^{k+1}(k!)}{10^k\,(k+1)!} = \lim_{k\to\infty} \frac{10}{k+1} = 0$$

$0 < 1$, so the series converges

19) Apply the ratio test:

$$\lim_{k\to\infty} \frac{a^{k+1}}{a^k} = \lim_{k\to\infty} \frac{3^{3(k+1)+1}}{5^{k+1}} \cdot \frac{5^k}{3^{3k+1}}$$

$$= \lim_{k\to\infty} \frac{3^3}{5} = \frac{27}{5} > 1$$

So, the series diverges.