PRACTICE PROBLEMS: MATH 105: SEC 2.5

#1) If $f: U \subset \mathbb{R}^n$ to $\mathbb{R}$ is differentiable, prove $g(\vec{x}) = f(\vec{x})^2 + 2f(\vec{x})$ is differentiable, and find the derivative in terms of $(Df)(\vec{x})$

Sol: note $f: U \to \mathbb{R}$ with U a subset of $\mathbb{R}^n$. Means f takes n pieces of input, gives one output (so it is scalar valued). In this case,

$$(Df)(\vec{x}) = \left(\frac{\partial f}{\partial x_1}(\vec{x}), \dots, \frac{\partial f}{\partial x_n}(\vec{x})\right)$$

For example, if $f(\vec{x}) = x_1 + 2x_2 + \dots + nx_n$, then

$$(Df)(\vec{x}) = (1, 2, \dots, n).$$

By Thm 10, page 151, the sum of differentiable fns is diff, so it suffices to show $f(\vec{x})^2$ and $2f(\vec{x})$ are diff. The first is diff as it is the product of two diff fns (again by Thm 10), and $2f(\vec{x})$ is a constant times a diff function.

We now use Thm 10 to compute the derivative:

↳ The derivative of $2f(\vec{x})$ is just $2(Df)(\vec{x})$. The deriv of $f(\vec{x})^2 = f(\vec{x})f(\vec{x})$, by the product rule, is just $(Df)(\vec{x})f(\vec{x}) + f(\vec{x})(Df)(\vec{x}) = 2f(\vec{x})(Df)(\vec{x})$

↳ Thus $(Dg)(\vec{x}) = 2f(\vec{x})(Df)(\vec{x}) + 2(Df)(\vec{x})$,

$$= 2(Df)(\vec{x}) * (f(x)+1)$$

#2e) Prove $f(x,y) = e^{xy} = \exp(xy)$ is differentiable, and find its derivative.

Soln: One approach is to use the chain rule.

Let $g(x,y) = xy$ so $g: \mathbb{R}^2 \to \mathbb{R}$

Let $f(x) = e^x$ so $f: \mathbb{R} \to \mathbb{R}$

Let $h(x,y) = f(g(x,y))$ so $h: \mathbb{R}^2 \to \mathbb{R}$

By Thm 11 (Chain Rule), h is differentiable if both g and f are, and this is readily checked.

<u>Alternate solution</u>: Let's calculate partial derivatives:

$$\frac{\partial f}{\partial x} = y e^{xy} \qquad \frac{\partial f}{\partial y} = x e^{xy}$$

Note the partial derivs exist and are continuous for all values of x and y. Thus By Thm 9 of Section 2.3 (pg 137), f is differentiable.

The derivative is thus

$$(\mathbf{D}f)(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)$$

$$= \left(y e^{xy}, x e^{xy}\right)$$

PRACTICE PROBLEMS! MATH 105: SEC 2.5 (CONT)

#11) Find $\left(\frac{\partial}{\partial s}(f \circ T)\right)(1,0)$ where $f(u,v) = \cos u \sin v$

and $T(s,t) = \left(\cos(t^2 s), \log\sqrt{1+s^2}\right)$

Soln: First, we can forget the chain rule and proceed by brute force.

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$f: \mathbb{R}^2 \to \mathbb{R}$

let $A: \mathbb{R}^2 \to \mathbb{R}$ be given by $A(s,t) = f(T(s,t))$

↳ Note that A is just $f \circ T$, the composition.

We have

$$
\begin{aligned}
A(s,t) &= f(T(s,t)) \\
&= f\left(\cos(t^2 s), \log\sqrt{1+s^2}\right) \\
&= \left[\cos\left(\cos(t^2 s)\right)\right] * \left[\sin\left(\log\sqrt{1+s^2}\right)\right] \\
&= \left[\cos(\cos(t^2 s))\right] * \left[\sin\left(\tfrac{1}{2}\log(1+s^2)\right)\right]
\end{aligned}
$$

To compute $\partial A/\partial s$ we simply use the product rule and many, many chain rules (though as t is fixed, these are 1-dim chain rules).

#11) CONTINUED

Using product / chain rules from Calc I, we find

$$\frac{\partial A}{\partial s} = -\sin\left(\cos(t^2 s)\right) * \left(-\sin(t^2 s)\right) * t^2 * \sin\left(\tfrac{1}{2}\log(1+s^2)\right)$$

$$+\cos(\cos(t^2 s)) * \cos\left(\tfrac{1}{2}\log(1+s^2)\right) * \frac{1}{2}\,\frac{2s}{1+s^2},$$

where we did <u>many</u> steps at once.

<u>Soln 2</u>: We use the multi-dimensional chain rule:

If $A(t,s) = f(T(s,t))$ then $(\mathbb{D}A)(t,s)$ equals
$(\mathbb{D}f)(T(s,t))\,(\mathbb{D}T)(s,t)$.

Now $(\mathbb{D}T)(s,t) = \,?$

well, $T(s,t) = \left(\cos(t^2 s), \tfrac{1}{2}\log(1+s^2)\right) = \left(T_1(t,s), T_2(t,s)\right)$

Thus
$$(\mathbb{D}T)(s,t) = \begin{pmatrix} \dfrac{\partial T_1}{\partial s} & \dfrac{\partial T_1}{\partial t} \\[2ex] \dfrac{\partial T_2}{\partial s} & \dfrac{\partial T_2}{\partial t} \end{pmatrix}$$

$$= \begin{pmatrix} -t^2 \sin(t^2 s) & -2ts\cos(t^2 s) \\[1ex] \dfrac{1}{2}\,\dfrac{2s}{1+s^2} & 0 \end{pmatrix}$$

#11) CONTINUED

$f(u,v) = \cos u \sin v$

Thus $(Df)(u,v) = \left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right)$

$= (-\sin u \sin v, \quad \cos u \cos v)$

so $(Df)(T(s,t)) = \big(-\sin T_1(s,t) \sin T_2(s,t),$

$\cos T_1(s,t) \cos T_2(s,t)\big)$

$= \Big(-\sin\big(\cos(t^2 s)\big) \sin\big(\tfrac{1}{2}\log(1+s^2)\big),$

$\cos(\cos(t^2 s)) \cos\big(\tfrac{1}{2}\log(1+s^2)\big)\Big)$

Now we just have to multiply the matrices at

$A = \cancel{\ } f \circ T$

$$\left(\frac{\partial A}{\partial s}, \frac{\partial A}{\partial t}\right) = \left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right)\Bigg|_{T(s,t)} \begin{pmatrix} \frac{\partial T_1}{\partial s} & \frac{\partial T_1}{\partial t} \\ \frac{\partial T_2}{\partial s} & \frac{\partial T_2}{\partial t} \end{pmatrix}\Bigg|_{(s,t)}$$

$$\text{so } \frac{\partial A}{\partial s} = \frac{\partial f}{\partial u}(T(s,t))\frac{\partial T_1}{\partial s}(s,t) + \frac{\partial f}{\partial v}(T(s,t))\frac{\partial T_2}{\partial t}(s,t)$$

Substituting gives the same answer as before, with u playing the role of T_1, v the role of T_2