MATH 105: PRACTICE PROBLEMS: SEC 2.5

Cousin of #4) Compute $\frac{\partial h}{\partial x}$ where $h = f(g(x,y))$ with $f(u,v) = \frac{v^2}{u^2+v^2}$, $u(x,y) = e^{-x-y}$, $v(x,y) = e^{xy}$

Soln: This is the example we did in class. The chain rule tells us that

$$(Dh)(x,y) = (Df)(g(x,y))\,(Dg)(x,y)$$

or

$$\left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right) = \left(\frac{\partial f}{\partial u}(g(x,y)), \frac{\partial f}{\partial v}(g(x,y))\right)\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

The matrix multiplication gives

$$\boxed{\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}(g(x,y))\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}(g(x,y))\frac{\partial v}{\partial x}}$$

This is the key equation

As $\frac{\partial f}{\partial u} = \frac{-v^2(2u)}{(u^2+v^2)^2} \Rightarrow \frac{\partial f}{\partial u}(g(x,y)) = \frac{-2e^{-x-y}e^{2xy}}{(e^{-2x-2y}+e^{2xy})^2}$

$\frac{\partial f}{\partial v} = \frac{2v(u^2+v^2) - v^2 2u}{(u^2+v^2)^2} \Rightarrow \frac{\partial f}{\partial v}(g(x,y)) =$ substitute

$\frac{\partial u}{\partial x} = -e^{-x-y}$ and $\frac{\partial v}{\partial x} = ye^{xy}$

substitute

We now substitute all of these into the formula for $\partial h/\partial x$, and that completes the analysis.

#5b) Verify the first special case of the chain rule when $f(x,y) = e^{xy}$ and $c(t) = (3t^2, t^3)$

Soln: Let $h(t) = f(c(t))$

Then $h(t) = f(3t^2, t^3) = e^{3t^2 t^3} = e^{3t^5}$

So $h(t) = e^{3t^5}$: we differentiate with the standard chain rule:

If $h(t) = A(B(t))$ then $h'(t) = A'(B(t))\,B'(t)$

here $A(t) = e^t$ $\quad B(t) = 3t^5$ (note $A(B(t)) = e^{3t^5}$)

$A'(t) = e^t$ $\quad B'(t) = 15t^4$

$A'(B(t)) = e^{3t^5}$

want to evaluate at $B(t)$, not t

$$\boxed{\text{Thus } h'(t) = e^{3t^5} \cdot 15t^4 = 15t^4 \cdot e^{3t^5}}$$

Turn over for second soln

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#5 b) Cont: $c(t) = (x(t), y(t)) = (3t^2, t^3)$

$$f(x,y) = e^{xy}$$

$$h(t) = f(c(t))$$

Soln using Chain Rule:

$$\frac{dh}{dt} = \frac{\partial f}{\partial x}(c(t))\frac{dx}{dt} + \frac{\partial f}{\partial y}(c(t))\frac{dy}{dt}$$

(note could also view as $(\nabla f)(c(t)) \cdot c'(t)$)

we have $\frac{\partial f}{\partial x} = ye^{xy} \rightarrow \frac{\partial f}{\partial x}(c(t)) = t^3 e^{3t^5}$

$\frac{\partial f}{\partial y} = xe^{xy} \rightarrow \frac{\partial f}{\partial x}(c(t)) = 3t^2 e^{3t^5}$

$\frac{dx}{dt} = (3t^2)' = 6t$

$\frac{dy}{dt} = (t^3)' = 3t^2$

Substituting yields

$$\boxed{\frac{dh}{dt} = t^3 e^{3t^5} \cdot 6t + 3t^2 e^{3t^5} \cdot 3t^2 = 15t^4 e^{3t^5},}$$

agreeing with the previous answer!