MATH 105: SOLUTION KEYS: HOMEWORK 5: DUE FEBRUARY 22, 2010

STEVEN MILLER

Section 2.2: #4: Show that $D = \{(x, y) : x \neq 0, y \neq 0\}$ is an open set.
Solution: This problem is very similar to the one in class, where we showed that $\{(x, y) : y > 0\}$ is open. What we must do is show that, no matter what point (x, y) in D that we are given, we can always find a radius r such that the ball of radius r about (x, y) is entirely contained in D; in other words, the ball does not hit either coordinate axis. We claim that if we take $r = \min(|x|/2, |y|/2)$ (the smaller of $|x|/2$ and $|y|/2$) then this works as the radius. To see this, we measure the distance from our point (x, y) to the coordinate axes. The distance to the x-axis is $|y|$, while the distance to the y-axis is $|x|$. Thus if we take our radius to be at most $|y|/2$, we see the ball cannot hit the x-axis. Similarly if we take the radius to be at most $|x|/2$ then the ball cannot hit the y-axis. By choosing r to be the minimum of these two, we ensure that we hit neither axis, and thus D is open.

Additional Problem #1: Let $f(x) = x^2 + 8x + 16$ and $g(x) = x^2 + 2x - 8$. Compute the limits as x goes to 0, 3 and ∞ of $f(x) + g(x)$, $f(x)g(x)$ and $f(x)/g(x)$.
Solution: We have $f(x)+g(x) = 2x^2+10x+8$ and $f(x)g(x) = x^4+10x^3+24x^2 - 32x - 128$. The limits at 0 and 3 are readily found for these as our functions are continuous, and are just $f(0) + g(0) = 8$, $f(3) + g(3) = 56$, $f(0)g(0) = -128$ and $f(3)g(3) = 343$. For $f(x)/g(x)$, note $g(x)$ is not zero for any of the points under consideration. Thus we may use the limit of a quotient is the quotient of the limits for 0 and 3, and find $f(0)/g(0) = -2$ and $f(3)/g(3) = 7$.

For the limits at ∞, we need to be a bit more careful. Looking at $f(x) + g(x)$, we see the $2x^2$ term dominates as $x \to \infty$, and thus the limit is ∞ (or, if you wish, undefined). For the product, we note that the main term is x^4, which tends to infinity as $x \to \infty$. Thus the limit of $f(x)g(x)$ as $x \to \infty$ is infinity again (or, if you wish, undefined again). For the last, the limit is actually 1. The easiest way to see this is to factor out x^2 from the numerator

Date: March 7, 2010.

and denominator, and then cancel:

$$\lim x \to \infty \frac{x^2+8x+16}{x^2+2x-8} = \lim_{x\to\infty} \frac{1+\frac{8}{x}+\frac{16}{x^2}}{1+\frac{2}{x}-\frac{8}{x^2}}.$$

As the limit of the numerator is 1 and the limit of the denominator is 1, we may use the limit of a quotient is the quotient of the limit, and find the answer is 1.

Additional Problem #2: Compute the derivative of $\cos(\sin(3x^2+2xlnx))$**. Note that if you can do this derivative correctly, you should be fine for the course.**

Solution: We may write this as $A(B(C(x)))$, with $C(x) = 3x^2 + 2x\ln x$, $B(x) = \sin x$ and $A(x) = \cos x$. The solution involves two chain rules, and then a product rule to evaluate $C'(x)$. The answer is $A'(B(C(x)))B'(C(x))C'(x)$. Evaluating everything yields

$$-\sin\left(\sin(3x^2+2xlnx)\right)\cdot\cos\left(3x^2+2x\ln x\right)\cdot(6x+2\ln x+2)\,.$$