MATH 105 SOLUTION KEYS

MURAT KOLOGLU

HOMEWORK 12

Section 2.6.

Exercise (23). The electrostatic potential V due to two infinite parallel filaments with linear charge densities λ and $-\lambda$ is $V = (\lambda/2\pi\epsilon_0)ln(r_2/r_1)$, where $r_1^2 = (x-x_0)^2+y^2$ and $r_2^2 = (x+x_0)^2+y^2$. We think of the filaments as being in the z direction, passing through the xy plane at $(-x_0, 0)$ and $(x_0, 0)$. Find $\nabla V(x, y)$.

Solution: The gradient of the function is given by $\nabla V(x, y) = (\partial V/\partial x, \partial V/\partial y)$. Differentiating with respect to x we have

$$\frac{\partial V}{\partial x} = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{r_2/r_1}\frac{\partial(r_2/r_1)}{\partial x}$$

Using the quotient rule to differentiate the last term we get

$$\begin{aligned}
\frac{\partial V}{\partial x} &= \frac{\lambda}{2\pi\epsilon_0}\frac{r_1}{r_2}\frac{(\partial r_2/\partial x)r_1 - r_2(\partial r_1/\partial x)}{(r_1)^2} \\
&= \frac{\lambda}{2\pi\epsilon_0}\frac{(\partial r_2/\partial x)r_1 - r_2(\partial r_1/\partial x)}{r_1 r_2} \\
&= \frac{\lambda}{2\pi\epsilon_0}\left(\frac{(\partial r_2/\partial x)}{r_2} - \frac{(\partial r_1/\partial x)}{r_1}\right)
\end{aligned}$$

Now, $r_1 = \pm\sqrt{(x-x_0)^2+y^2}$ and $r_2 = \pm\sqrt{(x+x_0)^2+y^2}$. Then (note that by dividing the partials of r_1 and r_2 by themselves, we get rid of the $\pm$ signs):

$$\frac{(\partial r_2/\partial x)}{r_2} = \frac{(x+x_0)}{(x+x_0)^2+y^2}$$

$$\frac{(\partial r_1/\partial x)}{r_1} = \frac{(x-x_0)}{(x-x_0)^2+y^2}$$

Thus

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$$\begin{aligned}\frac{\partial V}{\partial x} &= \frac{\lambda}{2\pi\epsilon_0}\left(\frac{(x+x_0)}{(x+x_0)^2+y^2} - \frac{(x-x_0)}{(x-x_0)^2+y^2}\right)\\ &= \frac{\lambda}{2\pi\epsilon_0}\left(\frac{(x+x_0)(r_1)^2-(x-x_0)(r_2)^2}{(r_1r_2)^2}\right)\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial V}{\partial y} &= \frac{\lambda}{2\pi\epsilon_0}\left(\frac{(\partial r_2/\partial y)}{r_2} - \frac{(\partial r_1/\partial y)}{r_1}\right)\\ &= \frac{\lambda}{2\pi\epsilon_0}\left(\frac{y}{(x+x_0)^2+y^2} - \frac{y}{(x-x_0)^2+y^2}\right)\\ &= \frac{\lambda}{2\pi\epsilon_0}y\left(\frac{((r_1)^2-(r_2)^2)}{(r_1r_2)^2}\right) \qquad (0.1)\end{aligned}$$

Combining these we have that

$$\nabla V = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{(r_1r_2)^2}\left\langle (x+x_0)(r_1)^2-(x-x_0)(r_2)^2, y((r_1)^2-(r_2)^2)\right\rangle .$$

Chapter 2 Review Exercises.

Exercise (47). The ideal gas law $PV = nRT$ *involves a constant* R*, the number* n *of moles of the gas, the volume* V*, the Kelvin temperature* T*, and the pressure* P*.*

(a) Show that each of n,P,T,V is a function of the remaining variables, and determine explicitly the defining equations.

Solution: Since each of n,P,T,V are variables, one can rearrange the ideal gas law as a function for any one of them. The explicit relations are:

$$n(P,V,T) = \frac{PV}{RT}$$

$$P(n,V,T) = \frac{nRT}{V}$$

$$T(n,V,P) = \frac{PV}{nR}$$

$$V(n,P,T) = \frac{nRT}{P}$$

(b) Calculate $\partial V/\partial T$, $\partial T/\partial P$, $\partial P/\partial V$ and show that their product equals -1.

Solution: Taking the required partials from the relations we found in (a):

$$\begin{aligned}\frac{\partial V}{\partial T} &= \frac{nR}{P}\\ \frac{\partial T}{\partial P} &= \frac{V}{nR}\\ \frac{\partial P}{\partial V} &= \frac{-nRT}{V^2}\end{aligned}$$

The product of these three partials is then

$$\begin{aligned}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P}\frac{\partial P}{\partial V} &= \frac{nR}{P}\frac{V}{nR}\frac{-nRT}{V^2}\\ &= \frac{-nRT}{PV}\\ &= \frac{-PV}{PV} \qquad \text{since } nRT = PV\\ &= -1.\end{aligned}$$

Section 3.1. *In Exercises 1 to 6, compute the second partial derivatives $\partial^2 f/\partial x^2$, $\partial^2 f/\partial x\partial y$, $\partial^2 f/\partial y\partial x$, $\partial^2 f/\partial y^2$ for each of the following functions. Verify Theroem 1 in each case.*

Exercise (1). $f(x,y) = 2xy/(x^2+y^2)^2$, on the region where $(x,y) \neq (0,0)$.
Solution: Taking the first partials we have

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{2y(x^2+y^2)^2 - 2xy(2(x^2+y^2)(2x))}{(x^2+y^2)^4}\\ &= \frac{2y\left((x^2+y^2)-4x^2\right)}{(x^2+y^2)^3}\\ &= \frac{2y\left(y^2-3x^2\right)}{(x^2+y^2)^3},\\ \frac{\partial f}{\partial y} &= \frac{2x(x^2+y^2)^2 - 2xy(2(x^2+y^2)(2y))}{(x^2+y^2)^4}\\ &= \frac{2x\left((x^2+y^2)-4y^2\right)}{(x^2+y^2)^3}\\ &= \frac{2x\left(x^2-3y^2\right)}{(x^2+y^2)^3}.\end{aligned}$$

Taking the second partials;

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{2y\,(y^2-3x^2)}{(x^2+y^2)^3}\right)\\
&= 2y\left(\frac{-6x(x^2+y^2)^3-(y^2-3x^2)(3)(x^2+y^2)^2(2x)}{(x^2+y^2)^6}\right)\\
&= 2y\left(\frac{-6x^3-6xy^2-6xy^2+18x^3}{(x^2+y^2)^4}\right)\\
&= 2yx\left(\frac{12x^2-12y^2}{(x^2+y^2)^4}\right)\\
&= \frac{24xy(x-y)(x+y)}{(x^2+y^2)^4},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{2x\,(x^2-3y^2)}{(x^2+y^2)^3}\right)\\
&= 2x\left(\frac{-6y(x^2+y^2)^3-(x^2-3y^2)(3)(x^2+y^2)^2(2y)}{(x^2+y^2)^6}\right)\\
&= 2x\left(\frac{-6y^3-6yx^2-6yx^2+18y^3}{(x^2+y^2)^4}\right)\\
&= 2yx\left(\frac{12y^2-12x^2}{(x^2+y^2)^4}\right)\\
&= \frac{24xy(y-x)(x+y)}{(x^2+y^2)^4},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{2y\,(y^2-3x^2)}{(x^2+y^2)^3}\right)\\
&= \frac{(6y^2-6x^2)(x^2+y^2)^3-2y(y^2-3x^2)(3)(x^2+y^2)^2(2y)}{(x^2+y^2)^6}\\
&= \frac{6y^4-6x^4-12y^4+36x^2y^2}{(x^2+y^2)^4}\\
&= \frac{-6y^4-6x^4+36x^2y^2}{(x^2+y^2)^4}\\
&= \frac{-6(x^4-6x^2y^2+y^4)}{(x^2+y^2)^4},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{2x\,(x^2-3y^2)}{(x^2+y^2)^3}\right)\\
&= \frac{(6x^2-6y^2)(x^2+y^2)^3-2x(x^2-3y^2)(3)(x^2+y^2)^2(2x)}{(x^2+y^2)^6}\\
&= \frac{6x^4-6y^4-12x^4+36y^2x^2}{(x^2+y^2)^4}\\
&= \frac{-6x^4-6y^4+36y^2x^2}{(x^2+y^2)^4}\\
&= \frac{-6(x^4-6x^2y^2+y^4)}{(x^2+y^2)^4}\\
&= \frac{\partial^2 f}{\partial y \partial x}.
\end{aligned}$$

With the last two we can see that $\partial^2 f/\partial x\partial y = \partial^2 f/\partial y\partial x$, and thus Theorem 1 is verified for f.

Exercise (8). Find all the second partial derivatives of
(a) $z = \sin(x^2 - 3xy)$
Solution: The first partial derivatives are

$$\frac{\partial z}{\partial x} = \cos(x^2-3xy)(2x-3y)$$
$$\frac{\partial z}{\partial y} = \cos(x^2-3xy)(-3x)$$

Thus the second partial derivatives are

$$\frac{\partial^2 z}{\partial x^2} = 2\cos(x^2-3xy) - \sin(x^2-3xy)(2x-3y)^2,$$
$$\frac{\partial^2 z}{\partial y^2} = -(3x)^2\sin(x^2-3xy),$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\big(\cos(x^2-3xy)(-3x)\big)\\
&= -\sin(x^2-3xy)(2x-3y)(-3x) - 3\cos(x^2-3xy)\\
&= 3x(2x-3y)\sin(x^2-3xy) - 3\cos(x^2-3xy).
\end{aligned}$$

By Theorem 1, $\partial^2 z/\partial x\partial y = \partial^2 z/\partial y\partial x$, thus we are done.

Exercise (11). Use Theorem 1 to show that if $f(x,y,z)$ is of class C^3, then

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial^3 f}{\partial y \partial z \partial x}$$

Solution: By Theorem 1 we know that for any C^2 function u

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

Since f is C^3, $\partial f/\partial z$ must be C^2. Then we can write (letting $u = \partial f/\partial z$;

$$\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2}{\partial y \partial x}\left(\frac{\partial f}{\partial z}\right)$$

We can rewrite this as

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x \partial z}\right)$$

But, again by Theorem 1, $\partial^2 f/\partial x \partial z = \partial^2 f/\partial z \partial x$. Substituting this into the equation we get;

$$\begin{aligned} \frac{\partial^3 f}{\partial x \partial y \partial z} &= \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial z \partial x}\right) \\ \frac{\partial^3 f}{\partial x \partial y \partial z} &= \frac{\partial^3 f}{\partial y \partial z \partial x}, \end{aligned}$$

exactly the expression we wanted to show.