HW #13
THREE SOLUTIONS GIVEN FOR EACH PROBLEM, USING THE METHOD / NOTATION OF THE BOOK, USING THE METHOD / NOTATION FROM CLASS, AND USING THE 'FAST' METHOD

Section 3.2 # 2,3 Determine the second-order Taylor formula for the given functions about the given points (x_0, y_0).

2)$f(x,y) = \frac{1}{(x^2+y^2+1)}$ and $x_0 = 0, y_0 = 0$.

Following book's method:

Note that $\frac{\delta f}{\delta x} = \frac{-2x}{(x^2+y^2+1)^2}$

$\frac{\delta f}{\delta y} = \frac{-2y}{(x^2+y^2+1)^2}$

$\frac{\delta^2 f}{\delta x^2} = \frac{2(3x^2-y^2-1)}{(x^2+y^2+1)^3}$

$\frac{\delta^2 f}{\delta y^2} = \frac{2(3y^2-x^2-1)}{(x^2+y^2+1)^3}$

$\frac{\delta^2 f}{\delta x \delta y} = \frac{8yx}{(x^2+y^2+1)^3}$

$\frac{\delta^2 f}{\delta y \delta x} = \frac{8xy}{(x^2+y^2+1)^3}$

So, $f(\mathbf{h}) = f(h_1, h_2) = f(x_0, y_0) + h_1(\frac{-2x}{(x^2+y^2+1)^2}) + h_2(\frac{-2y}{(x^2+y^2+1)^2}) + \frac{1}{2}(h_1^2(\frac{2(3x^2-y^2-1)}{(x^2+y^2+1)^3}) + h_2^2(\frac{2(3y^2-x^2-1)}{(x^2+y^2+1)^3}) + 2h_1h_2(\frac{8yx}{(x^2+y^2+1)^3})) + R_2(\mathbf{0}, \mathbf{h})$

By substituting in and simplifying we get:

$$f(\mathbf{h}) = 1 - h_1^2 - h_2^2 + R_2(\mathbf{0}, \mathbf{h})$$

Following method in class:

The second order Taylor expansion is given by

$$f(0,0) + (\nabla f)(0,0) \cdot (x,y) + \frac{1}{2}(x,y) \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(0,0) & \frac{\partial^2 f}{\partial x \partial y}(0,0) \\ \frac{\partial^2 f}{\partial y \partial x}(0,0) & \frac{\partial^2 f}{\partial y^2}(0,0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We now evaluate the derivatives at the origin:

$f(x,y) = 1/(x^2+y^2+1)$, so at the origin it is 1.

$\frac{\delta f}{\delta x} = \frac{-2x}{(x^2+y^2+1)^2}$, so at the origin it is 0.

$\frac{\delta f}{\delta y} = \frac{-2y}{(x^2+y^2+1)^2}$, so at the origin it is 0.

$\frac{\delta^2 f}{\delta x^2} = \frac{2(3x^2-y^2-1)}{(x^2+y^2+1)^3}$, so at the origin it is -2.

$\frac{\delta^2 f}{\delta y^2} = \frac{2(3y^2-x^2-1)}{(x^2+y^2+1)^3}$, so at the origin it is -2.

$\frac{\delta^2 f}{\delta x \delta y} = \frac{8yx}{(x^2+y^2+1)^3}$, so at the origin it is 0.

$\frac{\delta^2 f}{\delta y \delta x} = \frac{8xy}{(x^2+y^2+1)^3}$, so at the origin it is 0.

Thus the second order Taylor expansion is

$$1+(0,0)\cdot(x,y)+\frac{1}{2}(x,y)\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1+\frac{1}{2}(x,y)\begin{pmatrix} -2x \\ -2y \end{pmatrix} = 1-x^2-y^2.$$

Following the fast method in class:

We know the geometric series expansion, which gives

$$\frac{1}{1-r} = 1+r+r^2+\cdots.$$

If we take $r=-(x^2+y^2)$, then we have

$$\frac{1}{x^2+y^2+1} = \frac{1}{1-r}.$$

Expanding gives

$$\frac{1}{x^2+y^2+1} = 1-(x^2+y^2)+\cdots,$$

where we may ignore the next terms because they're of degree 4 and higher.

3) $f(x,y)=e^{x+y}$ and $x_0=0, y_0=0$

Note that:

$\frac{\delta f}{\delta x}=e^{x+y}$

$\frac{\delta f}{\delta y}=e^{x+y}$

$\frac{\delta^2 f}{\delta x^2}=e^{x+y}$

$\frac{\delta^2 f}{\delta y^2}=e^{x+y}$

$\frac{\delta^2 f}{\delta x\delta y}=e^{x+y}$

$\frac{\delta^2 f}{\delta y\delta x}=e^{x+y}$

So, $f(\mathbf{h})=f(h_1,h_2)=f(x_0,y_0)+h_1(e^{x+y})+h_2(e^{x+y})+\frac{1}{2}(h_1^2(e^{x+y})+h_2^2(e^{x+y})+h_1h_2(e^{x+y})+h_2h_1(e^{x+y}))+R_2(\mathbf{0},\mathbf{h})$

By substituting in and simplifying we get:

$$f(\mathbf{h})=1+h_1+h_2+\frac{1}{2}(h_1^2+h_2^2)+h_1h_2+R_2(\mathbf{0},\mathbf{h})$$

Following method in class:

The second order Taylor expansion is given by

$$f(0,0)+(\nabla f)(0,0)\cdot(x,y)+\frac{1}{2}(x,y)\begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(0,0) & \frac{\partial^2 f}{\partial x\partial y}(0,0) \\ \frac{\partial^2 f}{\partial y\partial x}(0,0) & \frac{\partial^2 f}{\partial y^2}(0,0) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

We now evaluate the derivatives at the origin; a straightforward computation shows that they are all equal to 1, which yields

$$\begin{aligned}
& 1+(1,1)\cdot(x,y)+\frac{1}{2}(x,y)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\\
= \quad & 1+x+y+\frac{1}{2}(x,y)\begin{pmatrix}x+y\\ x+y\end{pmatrix}\\
= \quad & 1+x+y+\frac{1}{2}\left(x(x+y)+y(x+y)\right)\\
= \quad & 1+x+y+\frac{1}{2}(x^2+2xy+y^2)\\
= \quad & 1+x+y+\frac{x^2}{2}+xy+\frac{y^2}{2}.
\end{aligned}$$

Following the fast method in class:

We have

$$e^x = 1+x+\frac{x^2}{2!}+\cdots, \quad e^y = 1+y+\frac{y^2}{2!}+\cdots.$$

Thus the fast method is to just multiply these together, and then only keep terms of degree 0, 1 or 2 (ie, 1, x, y, x^2, xy, y^2, and not terms like x^2y). Thus

$$\begin{aligned}
e^{x+y} &= e^x e^y\\
&= \left(1+x+\frac{x^2}{2!}+\cdots\right)\left(1+y+\frac{y^2}{2!}+\cdots\right)\\
&= 1+x+y+\frac{x^2}{2}+xy+\frac{y^2}{2}.
\end{aligned}$$