MATH 105 SOLUTION KEYS

MURAT KOLOGLU

HOMEWORK 20

Assume the probability that X equals x is $2e^{-2x}$ if $x \geq 0$ and 0 otherwise, and the probability that Y equals y is $3e^{-3x}$ if $y \geq 0$ and 0 otherwise. Show that both of these densities are, in fact, probability distributions (this means showing they are non-negative and integrate to 1), and calculate the probability that $X \geq Y$.

Solution: First note that the exponential function is always positive. Thus

$$2e^{-2x} \geq 0$$

$$3e^{-3x} \geq 0$$

for all x. Now (denoting $f(x)$ and $g(y)$ to be the probability distribution of x and y, respectively),

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \int_{0}^{\infty} 2e^{-2x}dx \\ &= \left(-e^{-2x}\right)_0^{\infty} \\ &= 0-(-1) \\ &= 1\end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} g(y)dy &= \int_{0}^{\infty} 3e^{-3y}dy \\ &= \left(-e^{-3y}\right)_0^{\infty} \\ &= 0-(-1) \\ &= 1.\end{aligned}$$

Therefore, the densities are indeed probability distributions.

Now to find the probability that $X \geq Y$, we want to condition y to be *at most* x. But x itself is not limited, it can take on any value. So the integral we are looking for is (assuming x and y are independent random variables);

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$$\int_{-\infty}^{\infty}\int_{-\infty}^{x} f(x)g(y)dydx$$

Another way of looking at this is arguing that we are after the volume under the joint distribution of x and y over the region on the xy plane where $x \geq y$, i.e. the triangle between the line $y = x$ and the x-axis (since the probabilities for negative values of x and y are 0).

Evaluating this integral gives us the probability that $X \geq Y$ to be

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{x} f(x)g(y)dydx &= \int_{0}^{\infty}\int_{0}^{x} (2e^{-2x})(3e^{-3y})dydx \\
&= \int_{0}^{\infty} (2e^{-2x})(-e^{-3y})_0^x dx \\
&= \int_{0}^{\infty} (2e^{-2x})(1 - e^{-3x})dx \\
&= \int_{0}^{\infty} (2e^{-2x} - 2e^{-5x})dx \\
&= \int_{0}^{\infty} 2e^{-2x}dx - 2\int_{0}^{\infty} e^{-5x}dx \\
&= \left(-e^{-2x}\right)_0^{\infty} - 2\left(\frac{-e^{-5x}}{5}\right)_0^{\infty} \\
&= 1 - (2)\left(\frac{1}{5}\right) \\
&= \frac{3}{5}.
\end{aligned}$$

Section 3 - Review Exercises.

Exercise (15). Find the points on the surface $z^2 - xy = 1$ nearest the origin.

Solution: One way of approaching this problem is by using Lagrange multipliers. The function we want to minimize is the square of the distance from the origin, $r^2(x, y, z) = x^2 + y^2 + z^2$ (we are minimizing the square of the distance to make the algebra simpler). The constraint we are minimizing on is $f(x, y, z) = z^2 - xy$. Then, by Lagrange's theorem we have;

$$\begin{aligned}
\nabla r^2(x, y, z) &= \lambda \nabla f(x, y, z) \\
\langle 2x, 2y, 2z \rangle &= \lambda \langle -y, -x, 2z \rangle
\end{aligned}$$

for some constant λ. Then we have the equations

$$\begin{aligned} 2x &= -\lambda y \\ 2y &= -\lambda x \\ 2z &= \lambda 2z \\ z^2 - xy &= 1 \end{aligned}$$

to solve. Substituting the second equation into the first one to eliminate λ we get,

$$\begin{aligned} 2\left(\frac{-2y}{\lambda}\right) &= -\lambda y \\ 4y &= \lambda^2 y \\ (4-\lambda^2)y &= 0 \end{aligned}$$

Then, either $\lambda = \pm 2$ or $y = 0$. If $\lambda = \pm 2$, $2z = \pm 4z$ implying $z = 0$, and $2x = \pm 2y$ implying $x = \pm y$. Plugging these into the constraint we get that $\pm x^2 = 1$ which means $x = \pm 1$ and thus $y = \mp 1$. But $r^2 = 2$ for these two points. Whereas, if $y = 0$, we have $x = 0$ and thus $z^2 = 1$ by the constraint. The solutions in this latter case are $(x, y, z) = (0, 0, \pm 1)$ which yield $r^2 = 1$ and thus are the solutions we are looking for.

Another way of approaching this problem is what might be termed the direct approach. We want to minimize $r^2(x, y, z) = x^2 + y^2 + z^2$ and we are given that

$$\begin{aligned} z^2 - xy &= 1 \\ z^2 &= 1 + xy. \end{aligned}$$

So, substituting the expression for z^2 into the function r^2 we get

$$r^2(x, y) = x^2 + y^2 + xy + 1.$$

Now, we know that this function has extrema at points for which the gradient is equal to zero. Then we are solving

$$(0, 0) = \nabla r^2(x, y) = (2x + y, 2y + x)$$

which gives us

$$\begin{aligned} 2x &= -y \\ 2y &= -x. \end{aligned}$$

Solving for x we get

$$\begin{aligned} 2x &= -\left(\frac{-x}{2}\right) \\ 4x &= x \\ 3x &= 0 \\ x &= 0 \end{aligned}$$

and thus $y = 0$. Once again we get that $z^2 = 1$, and thus the same set of solutions

$$(x, y, z) = (0, 0, \pm 1).$$