MATH 105 SOLUTION KEYS: HW #23

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HOMEWORK 23

Question: *Consider the surface*

$$(x/a)^2 + (y/b)^2 \ \le\ 1.$$

Find a change of variables to map this to a nice region, and then use that to find the area of the ellipse.

Solution: We let

$$u \ =\ \frac{x}{a}, \quad v \ =\ \frac{y}{b}$$

so the ellipse becomes the unit disk:

$$u^2 + v^2 \ \le\ 1.$$

We write down the change of variables explicitly:

$$T(x,y) \ =\ (u(x,y), v(x,y)) \ =\ \left(\frac{x}{a}, \frac{y}{b}\right)$$

or

$$T^{-1}(u,v) \ =\ (x(u,v), y(u,v)) \ =\ (au, bv)\,.$$

The derivative is

$$(DT^{-1})(u,v) \ =\ \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \ =\ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

and so the absolute value of the determinant of the derivative is

$$|\det((DT^{-1})(u,v))| \ =\ ab$$

(we'll assume a and b are positive). In other words, we have

$$dxdy \quad \longrightarrow \quad ab\, dudv.$$

By the change of variable formula, if we let E be the ellipse $(x/a)^2 + (y/b)^2 \le 1$ and $T(E)$ what the ellipse is mapped to, then $T(E)$ is just the unit disk D and

$$\int\int_E 1 dxdy \ =\ \int\int_{T(E)} 1ab\, dudv.$$

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As $T(E)$ is the unit disk, we find

$$\int\int_E 1 dx dy \;=\; ab \int\int_D 1 du dv.$$

Note that the integral of 1 over any region is just the area of that region, and thus the above becomes

$$\text{Area}(E) \;=\; ab\text{Area}(D).$$

We now see why this change of variables is so useful – it converts the integral for the ellipse's area to that of the unit disk, and we know that! The unit disk has area $\pi 1^2 = \pi$, as we thus finally obtain

$$\text{Area}(E) \;=\; \pi ab.$$

Note how miraculous this formula is – from knowing the circle's area we quickly get the area of an ellipse! This leads to the natural question: what is the volume of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 \leq 1$?