MATH 105: PRACTICE PROBLEMS FOR CHAPTER 2: SPRING 2010

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Question 1 : These problems deal with open sets.

(1) Let $S = \{(x,y,z) : 3x^2 + 4y^2 + 5z^2 < 6\}$. Is S open?
(2) Let $S = \{(x,y) : x^2 - y^2 = 1\}$. Is S open?
(3) Let $S = \{(x_1,\dots,x_n) : x_1^2 + \cdots + x_n^2 < 1\}$. Is S open?
(4) Let $S = \{(x,y,z) : x^2 + y^2 \le z\}$. Is S open?
(5) Let $S = \{(x,y) : xy = 1\}$. Is S open?
(6) Let $S = \{(x,y) : x^2 + y^2 > 1\}$. Is S open?

Question 2 : Compute the following limits (if they exist), or prove they do not. Remember $\log x$ means the logarithm of x base e.

(1) $\lim_{x\to 1}(x^4 - 2x^3 + 3x^2 + 4x - 5)$.
(2) $\lim_{x\to 2}\sin(3x^2 - 12)$.
(3) $\lim_{x\to 2}\frac{\sin(3x^2-12)}{x-2}$.
(4) $\lim_{x\to 0}\frac{\log x}{x}$.
(5) $\lim_{x\to 0}\frac{x}{\log x}$.
(6) $\lim_{(x,y)\to(0,0)}(4xy\cos(xy) + x^2 - y^3)$.
(7) $\lim_{(x,y)\to(0,0)}\frac{x^2y^2-1}{xy-1}$.
(8) $\lim_{(x,y)\to(1,1)}\frac{x^2y^2-1}{xy-1}$.
(9) $\lim_{(x,y)\to(0,0)}\frac{x^4-x^2y^2+y^4}{x^2+y^2+x^4y^4}$.
(10) $\lim_{(x,y)\to(0,0)} x^2y^3\cos\left(\frac{1}{x^2+y^2}\right)$.
(11) $\lim_{(x,y)\to(0,0)}\frac{x^2y^3\cos\left(\frac{1}{x^2+y^2}\right)}{x^2+y^2}$.
(12) $\lim_{(x,y,Z)\to(0,0,0)}\frac{x^3+y^3+z^3}{x^2+y^2+z^2}$.

Question 3 : Plot the level sets of value c for each function below (do enough values of c so you can recognize the result).

(1) $f(x,y) = \sin(x+y)$.
(2) $f(x,y) = (x+y)\sin(x+y)$.
(3) $f(x,y) = x^2 - 4y^2$.
(4) $f(x,y) = x^2 + 4y$.
(5) $f(x,y) = e^{\cos x}$.

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Question 4 : Find the gradients of the following functions:

(1) $f(x, y, z) = xy + yz + zx$.
(2) $f(x, y) = x\cos(y) + y\cos(x)$.
(3) $f(x_1, \dots, x_n) = x_1 x_2 \cdots x_n$.
(4) $f(x, y, z) = 1701x^{24601}\log(1793x^5y^4)$.
(5) $f(x, y) = \sin(x^2 + y^2)$.

Question 5 : Determine which functions below are differentiable.

(1) $f(x, y, z) = (xyz)^{4/3}$.
(2) $f(x, y) = (xy)^{2/3}$.
(3) $f(x_1, \dots, x_n) = (x_1 x_2 \cdots x_n)^2$.
(4) $f(x, y, z) = 1701x^{24601}\log(1793x^5y^4)$.
(5) $f(x, y) = \sin(x^2 + y^2)$.
(6) $f(x, y, z) = x^3\cos(x) + y^3\cos(y)$.
(7) $f(x, y) = xy\cos(1/y)$.

Question 6 : Find the tangent plane approximation to $f(x, y) = e^{xy} + 2\sin(x+y)\cos(x-y)$ at the point (x_0, y_0). Use the tangent plane plane to estimate $f(-.01, .02)$ by choosing (x_0, y_0) appropriately.

Question 7 : Parametrize the following curves, and find the tangent line approximation at the given point.

(1) A circle of radius 5 centered at $(2, 3)$ going counter-clockwise starting at the point $(7, 3)$; find the tangent line at the point $(7, 3)$.

(2) A circle of radius 5 centered at $(2, 3)$ going counter-clockwise starting at the point $(5, 7)$; find the tangent line at the point $(5, 7)$.

(3) The curve $y = e^x$ from $x = 1$ to $x = 10$; find the tangent line at the point $(2, e^2)$.

Question 8 : Consider the parametrized curve $c(t) = (\cos t, 2\sin t)$. What path does this trace out in the plane? Is it periodic (i.e., does it repeat where it is), and if so, what is the period? Does a particle whose position is given by $c(t)$ move at constant speed? If not, when is it moving fastest?

Question 9 : Find the derivative of $A(x, y, z) = (x^3y + y^2z + e^x)(\sin(y) + \cos(y) - z + 10)$.

Question 10 : Let $g(x, y, z) = (xy, yz, xz)$ and $f(u, v, w) = u^2 + v^2$. Set $A(x, y, z) = f(g(x, y, z))$. Compute DA.

Question 11 : Let $g(x, y, z) = xy^2z^3$. Compute the directional derivative of g at $(1, 1, 1)$ in the direction $\overrightarrow{v}$, where $\overrightarrow{v}$ is a unit vector in the direction $(3, 4, 12)$. In what direction is g increasing fastest?

Question 12 : Let $g(x, y, z) = e^x \cos \pi y + z \cos \pi x$. If possible, find the tangent plane to the level set of value 1 for $g(x, y, z)$ at $(x, y, z) = (1, 1, 1)$. If possible, find the tangent plane to the level set of value 0 for $g(x, y, z)$ at $(x, y, z) = (0, 1, 1)$.